Math 4140. Homework 6

due Wednesday, March 10

Note: All numbered sections, exercises, theorems and definitions are from Erdmann–Holm.

(1) Read Sections 3.3 and 3.4.

(2) Exercise 3.7.

(3) Exercise 3.11.

(4) Let A be a k-algebra, let V be an A-module, and let $\{e_1, e_2, \ldots, e_n\}$ be a set of orthogonal idempotents of A such that $e_1 + e_2 + \cdots + e_n = 1$. (Recall from Exercise 2.6 that "orthogonal" means that $e_i e_j = 0$ whenever $i \neq j$.) Prove that as a vector space, the space V admits the direct sum decomposition $V = e_1 V \oplus e_2 V \oplus \cdots \oplus e_n V$. Can the same decomposition always be viewed as a direct sum decomposition of modules?

(5) Let A be a k-algebra and let V be A-module. We say V is *indecomposable* if $V \neq 0$ and V cannot be written as a direct sum $V = M \oplus N$ of two nonzero submodules (in other words, if $V = M \oplus N$ then $M = 0$ or $N = 0$).

  (a) Show that every simple A-module is indecomposable.

  (b) Let $A = k[x]$, let $V = k^2$, and let V_α be the A-module whose underlying vector space is V and where x acts as the map $\alpha \in \mathrm{End}(V) = M_2(k)$ given by the matrix

$$\alpha = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

  (See Section 2.2 or Lecture 12 from February 12.) Show that V is indecomposable but not simple. Note that this example shows that the converse of the statement in (a) is false.