MATH 4140. HOMEWORK 2
Due Wednesday, February 3

Note: All numbered exercises are from Erdmann–Holm ([EH]).

(1) Read Sections 1.2 and 1.3 of [EH].

(2) Exercise 1.5.

(3) Exercise 1.6.

(4) Exercise 1.14.

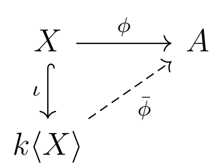
(5) Exercise 1.18.

Let k be a field below.

(6) Let $\varphi : A \to B$ be a k-algebra homomorphism. Show that $\ker \varphi$ is a two-sided ideal of A and $\operatorname{im} \varphi$ is a subalgebra of B.

(7) Prove that for any k-algebra A and any element $a \in A$, the evaluation map $\mathrm{Eval}_a : k[t] \to A, \sum \lambda_j t^j \mapsto \sum \lambda_j a^j$ is an algebra homomorphism.

(8) (Universal property of free algebras) Let X be a set, let $k\langle X \rangle$ be the free algebra of X over a field k, and let $\iota : X \to k\langle X \rangle$ be the map with $\iota(x) = x$ for all $x \in X$. Prove that for any algebra A and any function $\phi : X \to A$, there is a unique algebra homomorphism $\bar{\phi} : k\langle X \rangle \to A$ such that the following diagram commutes (i.e., such that $\phi = \bar{\phi} \circ \iota$).



(9) Let Q be the *two-loop quiver* with one vertex v and two distinct loops $\alpha : a \to a, \beta : a \to a$ at a. Show that the free algebra $k\langle X \rangle = \langle x, y \rangle$ on the set $X = \{x, y\}$ with two elements is isomorphic to the path algebra kQ. (*Hint*: Use the previous exercise to find a homomorphism, then argue that it is bijective.)