Math 4140. Lecture 41.

04. 28. 2021.

Last time: · properties of gp characters and char. tables. (e.g. S_4.)

Today: · Statement of the Krull-Schmidt Thm. ← read in more detail later, if you want.

· Course review.

1. Krull-Schmidt.

Let A be a k-algebra.

Def: (Def 7.1; see HW6). An A-module M is called <u>indecomposable</u> if it cannot be written as a direct sum $M = U \oplus V$ for nonzero submodules U, V. Otherwise, M is called <u>decomposable</u>.

(i.e., if $M = U \oplus V$ for submodules U, V, then either $U = 0$ or $V = 0$; i.e., no submodule has a nonzero "complement".)

Note / Recall that simple modules are certainly indecomposable.

The converse is not true in general, but if A is s.s, then an indecomposable module is necessarily simple by complete reducibility.

Thm. (Thms 7.5 & 7.18. but a bit more general: fin. dim $\to$ fin. length.)

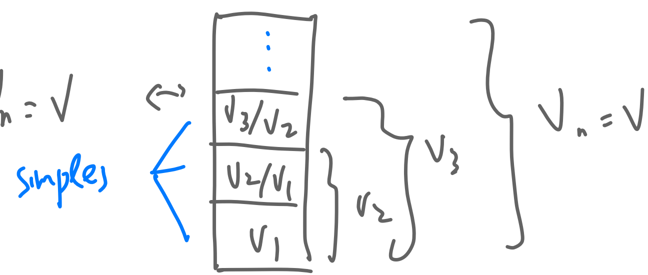
Let V be a left A-module of finite length. Then V can be decomposed as a finite direct sum of its indecomposable submodules. Moreover, the decomposition is unique in the sense that if $V = X_1 \oplus \cdots \oplus X_m$ and $V = Y_1 \oplus \cdots \oplus Y_n$ are two such decompositions, then $m = n$ and there is a permutation $\sigma \in S_n$ s.t. $X_i \cong Y_{\sigma(i)}$ $\forall\, 1 \le i \le n$.

- It's instructive to consider the Krull-Schmidt Thm. alongside the Jordan-Hölder Thm.

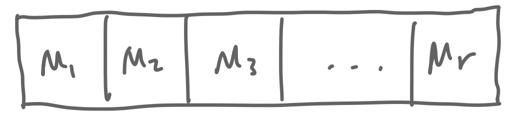
  — Both thms break down/decomp. finite-length A-modules V in some way.

  — The JH Thm breaks down V "vertically" using a filtration (the comp. series), where simple modules appear as the quotients of consecutive terms and hence may be viewed as the building blocks.

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V \quad \leftrightarrow$$



— The KS thm decomp. V "horizontally", into indecomposables:

$V_1 = M_1 \oplus M_2 \oplus \cdots \oplus M_r \leftrightarrow$ " 

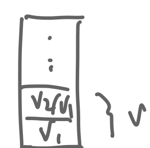
".

— If V (or A) is s.s, we may get both the JH and KS decompositions from the decomp. of V into simple modules.

$$\underbrace{V = L_1 \oplus \cdots \oplus L_r}_{\text{simples}} \quad \xrightarrow{\text{JH}} \quad V_i = L_1 \oplus L_2 \oplus \cdots \oplus L_i .$$

$$\xrightarrow{\text{KS}} \quad M_i = L_i .$$

In general, however, the decomps can be different in an essential way:

e.g., Ex 2.14 & 7.4. $T_n(k) \to$ indecomposable, but has length n. 

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2. Course review.

Some guidelines: (1) remember the definitions; know what to check,

e.g. - what is a gp algebra / path algebra ?

(what is it as a vec. space? how's mult. defined?)

- what do we need to do if we want to show
  - a set/vector space is an algebra
  - a map is an algebra/hom
  - a module is simple

(2). internalize the main theorems: iso thms for algebras and for modules, Jordan-Hölder, Schur's Lemma, that thm on Jacobson radicals, Artin-Wedderburn, Maschke.

Key topics by Chapter:

Ch 1:

- algebras, gp/path algebras, sub/quotient algebras. ideals, algebra homs.
- iso. thms for algebras.

Ch 2:

- modules and reps (and their equivalence), sub/quotient modules. module homs (common types: inclusions/projection)
- iso. thms for modules.

Ch 3:

- Simple modules, composition series, length of modules.
- The JH theorem, Lemma 3.18, Schur's Lemma.

Ch 4:

- semisimple modules, S.S. algebras, the Jacobson radical. nilpotent ideals, annihilators.
- That long theorem about Jacobson radicals.

Ch 5:

- opposite algebras, division algebras
- The statement of the A.W. theorem Cor 5.11. (significance of $r, n_1, \dots, n_r$).

Ch 6 :

- trivial rep, sign rep of S_n, def. of char.

- Maschke's Thm and its consequences, numerical deductions.
  (Thm 6.4, Cor 6.8.)

$\square$

THANK YOU FOR A FUN SEMESTER !!!