Math 4140. Lecture 40.

04. 26. 2021.

Let $k=\mathbb{C}$. Let G be a finite gp.

Last time:

- characters of a G-module V:

$$\rho: G \to GL(V) \quad \longrightarrow \quad \chi_V : G \to \mathbb{C},\ g \mapsto tr(\rho(g))$$

- characters are class functions in that they are constant on each conj. class of G.

- Notations and the char. table: $\mathbb{C}G \overset{A.W.}{\cong} \prod_{i=1}^{r} M_{n_i}(\mathbb{C})$

$M_{n_i}(\mathbb{C})$ acts via mult $\circlearrowright L_i$

$M_{n_j}(\mathbb{C})$ acts as 0 $\forall j \neq i$

$C_1 = \{e\}, C_2, \ldots, C_r$ are the conj. classes of G. Pick $g_i \in C_i$ representative

$n_1 = 1, n_2, \ldots, n_r$ .... dim. of the simple modules.

$L_1 = k_{triv}, L_2, \ldots, L_r$ .... simple modules

char table: (ij)-entry: $\chi_{L_i}(g_j)$

Today:

- Properties of characters

Properties of characters.

(1). Isomorphic G-modules afford the same characters.

Pf: Say V, W are iso. $\mathbb{C}G$-modules with a mod. iso $\varphi: V \to W$.

Then $\forall g \in G$, $\varphi(g\cdot v) \overset{*}{=} g\cdot\varphi(v) \quad \forall v \in V$.

Pick bases B, C for V and W, resp. Then * means

$$[\varphi]_B^C \cdot [g]_B \cdot v = [g]_C \cdot [\varphi]_B^C \cdot v \quad \forall v \in V$$

so

$$[\varphi]_B^C [g]_B = [g]_C [\varphi]_B^C$$

$$[g]_C = [\varphi]_B^C [g]_B \left([\varphi]_B^C\right)^{-1}$$

$$\Rightarrow \quad tr([g]_C) = tr([g]_B) \quad \Rightarrow \quad \chi_W(g) = \chi_V(g). \quad \square$$

(2). char. of a direct sum is the sum of the characters.

Pf: Let V, W be ($\mathbb{C}$)G-modules and consider their direct sum $U := V \oplus W$.

Pick a basis B for V and a basis C for W, then $B \cup C$ is a basis for $V \oplus W$, so for all $g \in G$,

$$[g_U]_{B\cup C} = \left[\begin{array}{c|c} [g_V]_B & 0 \\ \hline 0 & [g_W]_C \end{array}\right] \Rightarrow tr(g_U) = tr(g_V) + tr(g_W)$$

$$\Downarrow$$

$$\chi_U = \chi_V + \chi_W.$$

$\square$

Note: Since $\mathbb{C}G$ is s.s, the char χ_V of any G-module is in the span of $\chi_{L_1}, \chi_{L_2}, \dots, \chi_{L_r}$ by (1) and (2): $V \cong \bigoplus_{i=1}^{r} L_i^{m_i}$, some $m_i \in \mathbb{Z}_{\geq 0}$. $\Rightarrow \chi_V = \sum_{i=1}^{r} m_i \cdot \chi_{L_i}$.

3). The irreducible/simple characters $\chi_{L_1}, \chi_{L_2}, \dots, \chi_{L_r}$ form a basis of $C(G)$, the space of all class functions on G.

pf: We already noted that the functions $\delta_j : G \to \mathbb{C}$ defined by $\delta_j(g) = \begin{cases} 1 & \text{if } g \in C_j \\ 0 & \text{if } g \notin C_j \end{cases}$ where $1 \le j \le r$ form a basis of $C(G)$, so $\dim(C(G)) = r$. Thus it suffices to show that $\chi_{L_1}, \dots, \chi_{L_r}$ are linearly ind. This follows from the fact that for $e_j := Id_{M_{n_j}(\mathbb{C})} \in M_{n_j}(\mathbb{C})$ $(1 \le j \le r)$, we have

$\chi_{L_i}(e_j) = tr((e_j)_{L_i}) = \underline{\delta_{ij}} \cdot n_j$: if $\sum a_i \chi_{L_i} = 0$, then

Kronecker delta

$\sum a_i \chi_i(e_j) = 0 \ \forall j \Rightarrow a_j \cdot n_j = 0 \ \forall j \Rightarrow a_j = 0 \ \forall j.$ $\square$

Corollary: Two finite dimensional G-modules V and W are iso. iff $\chi_V = \chi_W$.

Pf: E.X.

(1)-(3) & the Cor.

The above four facts reveal why the character table is interesting:

Any char χ_V of G is a class function and hence is a unique linear comb. of the simple/irreducible characters. Moreover, if $\chi_V = \sum_{j=1}^{r} c_j \chi_{L_j}$, then we must have $V \cong \bigoplus_{j=1}^{r} L_j^{\oplus c_j}$.

E.g. We can determine how the regular module A of $A = \mathbb{C}S_3$ decomp. into simple modules using the char. table of S_3.

The table:

S_3	$g_1 = e$	$g_2 = (12)$	$g_3 = (123)$
$L_1 = k_{triv}$	1	1	1
$L_2 = k_{sgn}$	1	-1	1
$L_3 =$ "W"	2	0	-1

$(1, 1, 1)$

$(1, -1, 1)$

$(2, 0, -1)$

$[g_3]_{\{v_1 - v_2, v_2 - v_3\}} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$

$v_3 - v_1$

from midterm.

$\{ae_1 + be_2 + ce_3 : a+b+c = 0\} \subseteq \mathbb{C}^3$, the natural perm. module

χ_A	6	0	0

$(6, 0, 0)$

Linear algebra: $\chi_A = 1 \cdot \chi_{L_1} + 1 \cdot \chi_{L_2} + 2 \cdot \chi_{L_3} \Rightarrow \mathbb{C}S_3 \cong L_1 \oplus L_2 \oplus 2L_3$

as a $\mathbb{C}S_3$-module.

Properties of char. tables.

By the previous discussion, to understand (k) G-reps it is very useful to compute their character tables. Many tricks help:

(1) "obvious" simple reps: trivial rep k_{triv}, sign representations for S_n.

(2) the fact that $\chi_{L_i}(e) = \dim L_i = n_i$, which are sometimes known from numerical analysis of the A.W. decomp.

(3) tricks for creating simples, e.g. inflation of simples of quotients

(4) row and column orthogonal relations, e.g. $\sum_{i=1}^{r} \chi_i(g_j)\overline{\chi_i(g_k)} = 0$ if $j \neq k$

complex conj.

Example. S_4

S_4	e	(12)	(123)	(1234)	$(12)(34)$.	
L_1	1	1	1	1	1	← trivial rep.
L_2	1	-1	1	-1	1	← sign rep.
L_3	2	0	-1	0	2	← inflated from the dim 2 simple of S_3 via a hom $S_4 \twoheadrightarrow S_3$.
L_4	3	1	0	-1	-1	← the module $\{ae_1+be_2+ce_3+de_4 \mid a+b+c+d=0\}$ from HW. 6. Ex 6.5
L_5	3	-1	0	1	-1	← use ortho. relations