Math 4140. Lecture 39.

04. 23. 2021.

Last time: · more analysis of A.W. decomp $\mathbb{C}G \cong M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$ for G a finite gp.

In particular, $r = \#$ simple $\mathbb{C}G$-modules/iso $= \#$ conj. classes in G.

· HW discussion

Today: gp characters (associated to gp reps.)

Goal: to understand gp reps/modules via numerical data.

Group characters

Let G be a gp.

(1). Reps of a gp vs. reps of a gp algebra

Recall that (1) a rep. of a G is a vector space $\underline{V}$ equipped with
G-module
a gp hom $\rho: G \longrightarrow GL(V)$.

(2) a rep. of kG is, of course, a vec. space $\underline{V}$ equipped w/
kG-module
an algebra hom $\rho: kG \to \mathrm{End}_k(V)$.

(3) The data of a G-rep and a kG-rep are equivalent.

(1) $\to$ (2): extend linearly. (2) $\to$ (1): restrict

Today we'll use the language of gp reps.

12). Linear algebra

Let G be a finite gp and let $\rho: G \to GL(V)$ be a rep of G.

the natural mod.

running example: $G = S_3$, $k = \mathbb{C}$, $kG = \mathbb{C}S_3$, $V = \mathbb{C}\langle e_1, e_2, e_3 \rangle = \mathbb{C}^3$

the rep $\rho: G \to GL(\mathbb{C}^3)$ $\rho(g)(e_i) = e_{g(i)}$.

we'll consider two bases $B = \{e_1, e_2, e_3\}$

and $C = \{e_1+e_2+e_3, e_1-e_2, e_2-e_3\}$ of V

— For each $g \in G$, $\rho(G)$ is a linear map in $GL(V)$, so we can represent it as a matrix for every fixed basis of V.

— With respect to different bases of V, the matrix of $\rho(g)$ are generally different. e.g. $g = (1\,2) \in S_3$, then

$$[\rho(g)]_B = \begin{array}{c|c} & \begin{matrix} e_1 & e_2 & e_3 \end{matrix} \\ \hline \begin{matrix} e_1 \\ e_2 \\ e_3 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \quad \text{and} \quad [\rho(g)]_C = \begin{array}{c|c} & \begin{matrix} e_1+e_2+e_3 & e_1-e_2 & e_2-e_3 \end{matrix} \\ \hline \begin{matrix} e_1+e_2+e_3 \\ e_1-e_2 \\ e_2-e_3 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}.$$

— But the matrices of $\rho(g)$ w.r.t any two bases of V are always similar: indeed, recall that

$$[\rho(g)]_C = \underbrace{P_{C\leftarrow B}}_{\text{c.o.b. matrix.}} [\rho(g)]_B \, P_{B\leftarrow C} = P_{C\leftarrow B} [\rho(g)]_B \, P_{C\leftarrow B}^{-1}.$$

In particular, they have the same trace — indeed, this is the trace of the linear map $\rho(g)$ in linear algebra.

the sum of the diag. entries.

Def. (character of a rep.) Let G be a gp and V a G-module.

Let $\rho: G \to GL(V)$ be the corresponding rep. We define the character afforded by/of V to be the function $\chi_V : G \to k$ where

$\chi_V(g) = $ trace of $\rho(g)$ for all $g \in G$. e.g. $\chi_V((12)) = 1$.

— Note that the actions of conj. elts of G have the same trace:

if $g = xhx^{-1}$ for $g, h, x \in G$, then $tr(\rho(g)) = tr(\rho(h))$, i.e., $\chi_V(g) = \chi_V(h)$,

since $tr(\rho(g)) = tr(\rho(x)\rho(h)(\rho(x))^{-1}) = tr(\rho(h))$ → E.x.

$\chi_V((23))$
$=$
$\chi_V((13))$
$=$
1

Thus, for every G-rep V, χ_V is a class function in the sense that it is constant on every conj. class of G.

Def: We denote the ~~set~~ vec. space of all class functions on G by $C(G)$.

Some Computations :

- $\chi_V(e) = \operatorname{tr}([\rho(e)]_{\mathcal{B}}) = \operatorname{tr}([\mathrm{id}_V]_{\mathcal{B}}) = \operatorname{tr}(\underset{\dim V}{\underset{\shortparallel}{I_n}}) = \dim V.$ for any G-mod V.

- for the trivial module $V = \underset{\operatorname{span}\{1\}}{\underset{\shortparallel}{k}}$. $G \curvearrowright V$. $g \cdot 1 = 1 \Rightarrow [g]_{\{1\}} = [1]$,

$$\Rightarrow \chi_{triv}(g) = 1 \quad \forall g \in G.$$

- for the regular module $V = kG$, $G \curvearrowright kG$, $g \cdot h = gh = \begin{cases} h & \text{if } g = e \\ \neq h & \text{if } g \neq e. \end{cases}$

So $[\rho(g)]_{\{h : h \in G\}} = \begin{cases} I_{|G|} & \text{if } g = e \\ \text{a matrix w/ all-zero diag. entries} & \text{if } g \neq e \end{cases} \Rightarrow \chi_V(g) = \begin{cases} |G| & \text{if } g = e \\ 0 & \text{if } g \neq e. \end{cases}$

— A word on class functions on G:

From now on we assume that G has exactly r conj classes $C_1, C_2, \dots, C_r$. Let $\delta_i : G \to \mathbb{C}$ be defined by $\delta_i(g) = \begin{cases} 1 & \text{if } g \in C_i \\ 0 & \text{if } g \notin C_i \end{cases}$ $\forall 1 \le i \le r$.

Then clearly $\{\delta_i : 1 \le i \le r\}$ forms a basis of $C(G)$. e.g. a class function sending C_1 to 1, C_2 to 3 when $r=2$ is $1\cdot\delta_1 + 3\cdot\delta_2$.

$\updownarrow$

$(1, 3)$.

13). the character table of G.

Setup: G a finite gp. $\mathbb{C}G \overset{A.W.}{=} M_{n_1}(\mathbb{C}) \times \dots \times M_{n_r}(\mathbb{C})$

$e_1, e_2, \dots, e_r$: e_i is the id. in $M_{n_i}(\mathbb{C})$.

$L_1, L_2, \dots, L_r$ the simple G-mods up to iso, w/ $L_i \cong \mathbb{C}^{n_i}$.

$C_1, C_2, \dots, C_r$ are the conj. classes of G.

we'll assume that L_1 is the trivial module. $C_1 = \{e\}$.

Def: The character table of G is the $r \times r$ matrix where the (i,j) entry is $\chi_{L_i}(g_j)$ where g_j is any elt in C_j.

Thus, each row encode the character of a simple G-module, and each column tells the behavior of a conj. class in the diff simples.

e.g.

S_3	$g_1 = e$	$g_2 = (12)$	$g_3 = (123)$
k_{triv}	1	1	1
k_{sign}	1	-1	1
$\mathbb{C}\langle e_1 - e_2, e_2 - e_3 \rangle$	2	?	?

← E.X.

next time: prop. of char. tables.