Math 4140. Lecture 38.

Final: available 11:59 am May 1 → 11:59 pm May 2. 04. 21. 2021.

Last time: · more on the A.W. decomp of $\mathbb{C}G$. G a finite gp

So far: $\mathbb{C}G = M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$

$\Rightarrow$ · $r = \#$ simple modules of kG / up to iso

$= \#$ conj. classes of G.

· $n_1, n_2, \ldots, n_r$ = the dim.s of the simples

$|G| = \sum n_i^2$

· **Prop:** The number $N := |\{1 \le i \le r \mid n_i = 1\}|$ equals $|G/G'| = |G|/|G'|$ and hence divides $|G|$.

Today: · pf of the Prop. · gp characterers · HW discussion

Prop: (Corollary 6.8.) Let G be a finite gp. Then $N = |G/G'| = |G|/|G'|$.
In particular, N must divide $|G|$.

E.g. Example 6.10: $G = D_5$, $|G| = 10 \Rightarrow \mathbb{C}D_5 \cong \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C}) \times M_2(\mathbb{C})$.

$$\left(\begin{array}{l} 10 = N \cdot 1 + b \cdot 4 \\ N \mid 10 \end{array} \Rightarrow N = 2. \right)$$

Pf: Recall that there is a dim-preserving bijection between the sets (Lemma 2.43)

$$A = \{\text{modules of } G/G'\} \quad \text{and} \quad B = \left\{\begin{array}{l}\text{modules of } G \text{ on} \\ \text{which } G' \text{ acts as identity}\end{array}\right\}$$

It follows that $N = \#$ 1-dim (automatically simple) modules of G / iso

Claim: G' does act trivially on every 1-dim simple of G.

Pf: $\forall x, y \in G$, $[xy] \cdot v = xyx^{-1}y^{-1}v = \alpha\beta\alpha^{-1}\beta^{-1} \cdot v = v$ $\forall v \in V$ in a 1-dim simple V of G.

if $\left.\begin{array}{l} x \cdot v = \alpha v \\ y \cdot v = \beta v \end{array}\right\}$ has to be this way since V is 1-dim.

$\square$

It then further follows that

$$N = \#\text{ 1-dim modules of } G \text{ on which } G' \text{ acts as identity}$$

$$= \#\text{ 1-dim simples of } G/G' \quad (\text{by the bijection } A \Leftrightarrow B)$$

$$= \#\text{ simples of } G/G', \text{ the "}r\text{" for } \mathbb{C}(G/G') \quad (\text{since } G/G' \text{ is abelian})$$

$$= |G/G'| \quad \left(\text{by the formula } |H| = \sum_{i=1}^{r} n_i^2 \text{ for } H = \prod_{i=1}^{r} M_{n_i}(\mathbb{C})\right)$$

$\square$

HW 10.

6.4. $G = S_3$, $A = \mathbb{C}G$, $\sigma = (1\,2\,3)$ $\tau = (1\,2)$.

(a). $e_\pm := \frac{1}{6}(1 \pm \tau)(1 + \sigma + \sigma^2)$ $\to$ show they are orthogonal idemp.

$\tau\sigma = (1\,2)(1\,2\,3) = (1)(2\,3)$ $\to \mathrm{sgn}(\tau\sigma) = -1$

$\tau\sigma^2 = \tau\sigma^2 = (1\,2)(1\,3\,2) = (1\,3) \to \mathrm{sgn}(\tau\sigma^2) = 1$.

Computation $\longrightarrow$ · $e_+ = \frac{1}{6}\sum_{g \in G} g =: w$ from the proof of Mashke's theorem.

$h \cdot w = w$.

· $e_- = \frac{1}{6}\sum_{g \in G} \mathrm{sgn}(g)\, g$

$\downarrow$

$\mathrm{sgn}(gh)$ since sgn is a rep.

a typical term out of the "36" in e_-^2 : $\underbrace{\mathrm{sgn}(g)\mathrm{sgn}(h)}_{} gh$.

(b). ✓

(c). Work out the mult. table for the four elts.

Note that no computation w/ permutations is needed:

$$f\tau f_1 = f\tau\tau f\tau^{-1} = ff\tau^{-1} = f\tau^{-1} = f\tau$$

$$f\tau f = f\tau f\tau^{-1}\tau = ff_1\tau = 0\tau = 0$$

(d). We know that the three matrix alg. should be $\mathbb{C}$, $\mathbb{C}$, $M_2(\mathbb{C})$.

hope: $\mathbb{C} \leftrightarrow e_+$, $\mathbb{C} \leftrightarrow e_-$, $M_2(\mathbb{C}) \overset{(c)}{\updownarrow} \{f, f\tau, \tau f, f_1\}$.

E_{11} E_{12} E_{21} E_{22}

$\mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$

$(1, 0, 0) \overset{?}{\leftrightarrow} e_+$

$(0, 1, 0) \leftrightarrow e_-$

$(0, 0, E_{ij}) \leftrightarrow$ Part (c) stuff.

To make this precise, we could

- show that $B = \{e_+, e_-, f, f\tau, \tau f, f_1\}$ is a basis of $\mathbb{C}G$.

  has "correct dim": 6, so it suffices to prove B is lin. ind.

orthogonality helps.

Some orthogonality conditions: $e_+ f = 0$?

$$f + f_1 \overset{(*)}{=\!=} 1 - e_- - e_+ \Rightarrow e_+ f + e_+ f_1 = e_+ - 0 - e_+ = 0 \Rightarrow e_+ f f + e_+ f_1 f = 0$$

$$\Downarrow$$

$$e_+ f = 0$$

The linear ind (after finding enough orthogonal pairs like e_+, f):

Say $a_1 e_+ + a_2 e_- + a_3 f + a_4 f\tau + a_5 \tau f + a_6 f_1 = 0$, want $a_i = 0 \ \forall i$.

e.g. $e_+ \cdot \mathrm{LHS} = 0 \Rightarrow a_1 = 0$.

· establish a bijection $\mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C}) \leftrightarrows \mathbb{C}G$

by finding a linear iso and showing it respects mult.

6.7. (a). Recall that if G is abelian, then

$$\mathbb{C}G = \underbrace{\mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C}}_{|G|}.$$

(b). Use numerics. $\sum n_i^2 = 8$. $\Rightarrow$ $n_i \leq 2 \quad \forall i$

non abelian $\Rightarrow$ not all n_i are 1.

also, at least one n_i is 1. (trivial rep)

6.9. (i). no copy of $\mathbb{C}$ ...

(ii). prove/invoke the fact that $|G|$ is a prime number,

then G is cyclic and hence abelian.

(iii). S_3. ✓

6.5. $\underbrace{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{bmatrix}}_{\text{perm.}} \cdot (\underbrace{10, 1, 5, 2}_{10e_1+e_2+5e_3+2e_4}) = (\underbrace{5, 10, 1, 2}_{5e_1+10e_2+e_3+2e_4})$

need to show V is simple. $\to$ same strategy as always:

take an arbitrary $0 \neq v \in V$. show that $\underbrace{\mathbb{C}S_n}_{A} \cdot v = V$.

Note: The set $C = \{ \underline{e_i} - e_j \mid 1 \le i, j \le n, i \ne j \}$ spans V.

standard basis elt

(indeed $\{e_1 - e_2, e_2 - e_3, \cdots, e_{n-1} - e_n\}$ is a basis of V)

Moreover, if we have any elt $w \in C$, then we can get

all elts in C via the A-action

e.g. $n=5$, $\quad e_3 - e_4 \longrightarrow \begin{array}{c} \quad\;\; 3 \;\; 4 \\ \left[\cdots \;\; i \;\; j \;\; \cdots \right] \end{array} \cdot (e_3 - e_4) = e_i - e_j \quad \forall i, j.$

$V = A \cdot v \Leftarrow$ we can get some particular $e_i - e_j$ by letting

A act on v.

e.g. best case: the random $0 \ne v \in V$ you selected is already a multiple of $v_i - v_j$ for some i, j.

$5e_1 - 5e_3$
$||$
$(5, 0, -5, 0, \cdots, 0)$

What happens if your v isn't $\lambda(e_i - e_j)$?

$\Downarrow$

v has at least 3 nonzero entries, and at least two of them (i, j) are diff.

because the coordinates have to sum to 2.

e.g. $v = (\overset{i}{3}, 0, 0, \overset{j}{5}, -2, -6, 0)$

$(ij) = (14)$.

$(ij)\cdot v = (5, 0, 0, 3, -2, -6, 0)$

$(ij)\cdot v - v = (2, 0, 0, -2, 0, 0, \cdots)$

$= 2(e_i - e_j)$. → back in case 1.

To write the actual proof, discuss the two cases carefully.