Math 4140. Lecture 37.

04. 19. 2021.

Last time: A.W. decomp of s.s. f.d. gp algebras:

$$\mathbb{C}G \overset{*}{=} M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$$

$\Rightarrow$ · $r = \#$ iso classes of simple modules of kG

· $n_1, n_2, \ldots, n_r$ are the dimensions of these simple modules

· $|G| = \sum_{i=1}^{r} n_i^2$

$g \cdot v = v \quad \forall g \in G, v \in V$

· Since $\mathbb{C}G$ always has the trivial module $\circlearrowleft V_{tri} = \mathbb{C}$

we may assume $n_1 = 1$, corresponding to V_{tri}.

Today: More facts about $(*)$: "$r = \#$ conj. classes of G"; one dim. simples.

$r = \#$ conj. classes in G.

$g, h \in G$ are conj if $h = x g x^{-1}$ for some $x \in G$.
$(g \sim h) \to$ equiv. rel. $\to$ equiv. classes

Thm: In $*$ (when it exists in the setup of Page 1), the number r also equals the number of conj. classes of G.

$$\mathbb{C}G \overset{*}{\cong} M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$$

Pf idea: double counting $\dim(\underset{\text{center}}{Z}(\mathbb{C}G))$.

On the one hand,

$$\dim(Z(\mathbb{C}G)) = \dim\Big(Z\big(M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})\big)\Big)$$ (since $\mathbb{C}G \cong M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$)

$(a_1, a_2, \ldots, a_r)$

$$= \dim\big(Z(M_{n_1}(\mathbb{C})) \times \cdots \times Z(M_{n_r}(\mathbb{C}))\big)$$ (follows from def. of centers)

$Z(M_n(\mathbb{C})) = \{\lambda \cdot \mathrm{Id}_n \mid \lambda \in \mathbb{C}\} \cong \mathbb{C}$.

$$= \dim\big(\underset{\cong \mathbb{C}}{\{\lambda \cdot \mathrm{Id}_{n_1}\}} \times \cdots \times \underset{\cong \mathbb{C}}{\{\lambda \cdot \mathrm{Id}_{n_r}\}}\big)$$ (by explicit computation)

$a \in \mathbb{C}$

$E_{ij} \cdot E_{kl} = \delta_{jk} E_{il}$ Ex. 3.16.

$$= r$$

On the other hand, we have: $\boxed{gh = hg \iff hgh^{-1} = g}$

Prop: (Prop 6.11.) Let G be a finite gp and let k an arbitrary field. Let $\mathcal{C}$ be the set of conjugacy classes of G. Then the class sums $\left\{ \underline{C} := \sum_{g \in C} g \;\middle|\; C \in \mathcal{C} \right\}$ is a k-basis of $Z(kG)$.

pf: We first show that $\underline{C} \in Z(kG)$ $\forall C$:

$$h \underline{C} h^{-1} = h\left(\sum_{g \in C} g\right) h^{-1} = \sum_{g \in C} (hgh^{-1}) \overset{x = hgh^{-1}}{=\!=} \sum_{x \in C} x = \underline{C} \qquad \forall h \in G.$$

since conj. by h defines a bijection from C to C. So $\underline{C} \in Z(kG)$.

Linear independence ("trivial"): Say $\mathcal{C} = \{C_1, C_2, \dots, C_k\}$

$$\sum_{i=1}^{k} a_i \underline{C_i} = 0 \Rightarrow \overbrace{a_1(\underbrace{g_1^{(1)} + \cdots}_{\text{elts in } C_1}) + a_2(\underbrace{g_1^{(2)} + g_2^{(2)} + \cdots}_{\text{elts in } C_2}) + a_k(\underbrace{g_1^{(k)} + \cdots}_{\text{elts in } C_k})}^{\text{in } kG} \Rightarrow a_i = 0 \;\; \forall i$$

since $\{g \in G\}$ is lin. ind. in kG.

"spanning": Take an arbitrary elt $w \in Z(kG)$. Write $w = \sum_{x \in G} \alpha_x x$.

It suffices to show that $\alpha_y = \alpha_{gyg^{-1}}$ $\forall g \in G$. $\forall y \in G$.

Take any $g \in G$, since $w \in Z(kG)$, we have $gwg^{-1} = w$.

But $gwg^{-1} = g \sum_{x \in G} \alpha_x x\, g^{-1} = \sum_{x \in G} \alpha_x\, gxg^{-1} \overset{y = gxg^{-1}}{\underset{x = g^{-1}yg}{=\!=}} \sum_{y \in G} \alpha_{g^{-1}yg}\, y$

So $\underbrace{\sum_{y \in G} \alpha_{g^{-1}yg}\, y}_{①} \overset{gwg^{-1} = w}{=\!=\!=} \sum_{x \in G} \alpha_x x = \underbrace{\sum_{y \in G} \alpha_y y}_{②}$.

Comparing the coeffs of ① and ②, we get that $\alpha_y = \alpha_{g^{-1}yg}$ $\forall g \in G$. $\forall y \in G$.

It follows that $\alpha_y = \alpha_{gyg^{-1}}$ $\forall g \in G$, $\forall y \in G$, as desired. $\square$.

<u>Corollary</u>. The theorem holds, with $r = \dim Z(kG) = |\mathcal{C}|$.

Number of one-dimensional simples

We are interested in the number $N := \left| \{ n_i \mid 1 \le i \le r, \ n_i = 1 \} \right|$.

(Recall that we already know $N = r \Leftrightarrow kG$ is comm. $\Leftrightarrow G$ is abelian.)

Def: The <u>commutator subgroup</u> G' of a gp G is the subgp generated by elts of the form $[xy] := xyx^{-1}y^{-1}$ where $x, y \in G$.

Recall/Note: (i). G' is normal in G.

(ii). Let N be a normal subgp of G. Then G/N is abelian $\Leftrightarrow G' \le N$.

$\left(G/N \text{ is abelian} \Leftrightarrow xNyN = yNxN \ \forall x,y \in G \Leftrightarrow xy(yx)^{-1} = xyx^{-1}y^{-1} \in N \ \forall x,y \in G \right)$

i.e., $xyN = yxN \ \forall x,y \in G$

Prop: (Corollary 6.8.) Let G be a finite gp. Then $N = |G/G'| = |G|/|G'|$.

In particular, N must divide $|G|$.

E.g. Example 6.10: $G = D_5$, $|G| = 10 \Rightarrow \mathbb{C}D_5 \cong \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C}) \times M_2(\mathbb{C})$.

$$\left(\begin{array}{l} 10 = N \cdot 1 + b \cdot 4 \\ N \mid 10 \end{array} \Rightarrow N = 2. \right)$$

Pf: next time.