Math 4140. Lecture 36.

04.16.2021.

Last time: • Maschke's Thm: Let k be a field and G is finite gp.

Then kG is s.s. iff $\mathrm{char}(k) \nmid |G|$.

Main Ideas for the proof:

- "only if": if $\mathrm{char}(k) \mid |G|$, then the submodule $U := \mathrm{Span}\langle \sum_{g \in G} g \rangle \subseteq kG$ cannot have a complement.
- "if": — show kG is completely reducible if $\mathrm{char}(k) \nmid |G|$
  — for any submodule $W \subseteq kG$, find a complement: use a special map $\pi : kG \to W$.

Today: • finish the proof, by showing that Φ is a kG-mod. hom.

• Some consequences of the A.W. thm + Maschke's Thm.

1. Proof that π is an module hom.

$$\pi : kG = W \oplus \underline{\underline{V}} \longrightarrow W \quad , \quad m \longmapsto \frac{1}{|G|} \sum_{g \in G} g \cdot \underline{\underline{p}}(g^{-1} \cdot m)$$

V.s. complement of W in kG

natural projection $p : kG \to W$ 'from linear algebra'

Pf that Φ is a kG-module hom:

$$\boxed{x = hg \Rightarrow g = h^{-1}x \Rightarrow g^{-1} = x^{-1}h}$$

linearity. ✓ see last lecture

π respects the kG-action: we need to show that $\pi(h \cdot m) = h \cdot \pi(m)$ $\forall m \in kG, h \in G$.

$$RHS = h \cdot \pi(m) = h \cdot \frac{1}{|G|} \sum_{g \in G} g \, p(g^{-1} m) = \frac{1}{|G|} \sum_{g \in G} hg \, p(\underline{\underline{g^{-1}}} m) \overset{x := hg}{=\!=} \frac{1}{|G|} \sum_{x \in G} x \, p(x^{-1} h \cdot m)$$

$$LHS = \pi(h \cdot m) = \frac{1}{|G|} \sum_{g \in G} g \, p(g^{-1} \cdot h \cdot m) = \frac{1}{|G|} \sum_{g \in G} g \, p((g^{-1}h) m)$$

$\square$

2. Consequences of Maschke's Thm.

From now on we focus on the field $k = \mathbb{C}$ which is alg. closed and has char. 0.

Thm 6.4. Let G be a finite gp. Then $\mathbb{C}G$ is s.s and therefore has an A.W decomp

$$\mathbb{C}G \cong M_{n_1}(\mathbb{C}) \times M_{n_2}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C}).$$

Moreover, (a). The gp alg. $\mathbb{C}G$ has precisely r simple modules up to iso;
the dimensions of these modules are $n_1, n_2, \ldots, n_r$.

(b). We have $|G| = \sum_{i=1}^{r} n_i^2$.

(c). The gp G is abelian iff all simple $\mathbb{C}G$-modules have dim 1.

Fact (Thm 6.12): The number r above also equals the number of conjugacy classes of G. ← will be proved next time.

Pf: (a). " $r = \#$ iso classes of simples of $\Bbbk G$ " : part of the A.W. theorem.

(Cor. 5.11.)

(b). " $|G| = \sum_{i=1}^{r} n_i^2$ " : take the dimensions of both sides of

$$\Bbbk G \cong M_{n_1}(\mathbb{C}) \times M_{n_2}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$$

(c). " G is abelian $\Leftrightarrow$ $n_i = 1 \ \forall i$ " :

G is abelian $\Leftrightarrow$ $\Bbbk G$ is comm. $\Leftrightarrow$ $M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$ is comm.

$\Leftrightarrow$ $M_{n_i}(\mathbb{C})$ is comm. $\forall i$

$\Leftrightarrow$ $n_i = 1 \quad \forall\, 1 \le i \le r$. $\square$

3. AW-decomp of $\mathbb{C}S_3$.

What does Thm 6.4 imply about the A.W. decomp of $\mathbb{C}S_3$?

+ our midterm

Say it's $\quad \mathbb{C}S_3 \overset{*}{=} M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$

- We know each class of simple modules of $\mathbb{C}S_3$ gives rise to a factor $M_{n_i}(\mathbb{C})$ in $(*)$.

- We also know that $\mathbb{C}S_3$ has a trivial module U and 2-dim simple module W appearing as submodules of the natural module $V = \mathbb{C}\langle e_1, e_2, e_3\rangle$, with $U = \mathrm{Span}\langle v_1+v_2+v_3\rangle$, $W = \mathrm{Span}\langle v_1-v_2, v_2-v_3\rangle$, so up to reordering we may assume that $r \geq 2$ and $n_1 = 1$, $n_2 = 2$.

· By Thm 6.4, we have $6 = |S_3| = n_1^2 + n_2^2 + \cdots + n_r^2$

$= 1^2 + 2^2 + \cdots + n_r^2$

$= 5 + \cdots + n_r^2$

It follows that we have to have $r = 3$ and $n_3 = 1$; this n_3 must correspond to a simple $\mathbb{C}S_3$-module which has dim 1 and is not iso. to the trivial module $\mathcal{U}$.

What's that third simple $\mathbb{C}S_3$-module?

<u>A</u>: It's the sign module $S = \mathbb{C}$ with the action $g \cdot 1 = \mathrm{sgn}(g)\ \forall g \in S_3$.

(this module corresponds to the gp rep $S_3 \to GL(\mathbb{C})$, $g \mapsto \mathrm{sgn}(g)$.)

Recall: any symm. gp S_n has a sign module of dim 1 defined this way.

What about $\mathbb{C}S_4$?

$$\mathbb{C}S_4 \overset{*}{=} \prod_{i=1}^{r} M_{n_i}(\mathbb{C})$$

Why 1, 1, 2, 3, 3 ?
What exactly are the simples?

look up Young tableaux & Specht modules!

- $24 = |S_4| = \sum_{i=1}^{r} n_i^2$
- We know $r \geq 2$ because there should be two factors in $*$ corresponding to the trivial module and the sign module of $\mathbb{C}S_4$; moreover, they have dim 1, so we can assume $n_1 = n_2 = 1$.
- what's r ? We can use Thm 6.12 and gp theory.

$$r = \#\text{ conj classes in } S_4 = \#\text{ cycle types in } S_4. = 5.$$

- So: $24 = 1^2 + 1^2 + n_3^2 + n_5^2$

$$22 = n_3^2 + n_4^2 + n_5^2 \Longrightarrow \{n_3, n_4, n_5\} = \{2, 3, 3\} \text{ by numerical considerations.}$$

$\square$