Math 4140. Lecture 35.

04.14.2021.

Last time: · finished the Artin-Wedderburn Thm: Structure of s.s. algebras, simple modules of s.s. algebra, uniqueness.

Today: · We'll prove Maschke's Thm:

Thm: (Thm 6.3.) Let K be a field and let G be a finite gp. Then the gp algebra kG is s.s. iff the characteristic $\underline{\text{char}(k) \text{ of } k \text{ does not divide } |G|}$.

$\Updownarrow$

" $|G| = 0$ in k "

Note an immediate corolary:

If $\text{char}(k) = 0$ (e.g. $k = \mathbb{Q}$, $k = \mathbb{R}$, $k = \mathbb{C}$) then kG is s.s. for all finite gps G. (so we always get an A.W decomposition → interesting)

The "only if" direction.

We'll prove that if kG is s.s then $\mathrm{char}(k) \nmid |G|$

by considering the complete reducibility of the regular module kG.

Key object: the elt $w := \sum_{g\in G} g \in kG$ (finite sum, makes sense) since $|G| < \infty$.

Note: · $\forall h \in G$, we have $h \cdot w = \sum_{g\in G} hg = \sum_{x\in G} x = w$ since the map $l_h : G \to G$, $g \mapsto hg$ is a bijection.

· Consequently, the subspace $U := \mathrm{Span}\{w\}$ is a submodule of kG and $w \cdot w = \left(\sum_{g\in G} g\right) \cdot w = \sum_{g\in G} (g\cdot w) = \sum_{g\in G} w = |G| w$.

Pf: Assume kG is s.s, then kG is c.r. so the submodule U has a complement C, ie., a submodule C of kG s.t. $kG = U \oplus C$ as modules. In particular, $1_{kG} = \lambda w + c$ for some $\lambda \in k$ and $c \in C$.

Note that $\lambda \neq 0$: otherwise $1 = 0 + c = c \in C$, so $g = g \cdot 1 \in C \Rightarrow kG \subseteq C$,

Contradiction.

But then $w = w \cdot 1 = w(\lambda w + c) = \lambda w^2 + w \cdot c = \lambda |G| w + w \cdot c$

so $wc = w - \lambda |G| w = (1 - \lambda |G|) \cdot w \in U \cap C = 0$.

It follows that $\lambda |G| \neq 0$, so $|G| \neq 0$ (in k). so $\mathrm{char}(k) \nmid |G|$.

Rmk: It might be more natural to prove the contrapositive:

" kG s.s $\Rightarrow \mathrm{char}(k) \nmid |G|$ " $\longrightarrow$ " if $\mathrm{char}\, k \mid |G|$, then kG is not s.s."

by showing that if $\mathrm{char}(k) \mid |G|$ then the submodule $U \leq kG$ has no complement, so kG is not c.r. and hence not s.s.

The "if" direction:

We will show that kG is s.s. if $\mathrm{char}(k) \nmid |G|$ by proving that kG is completely reducible.

Preparation: Recall from the midterm ~~that~~ if $j: N \to M$ and $\pi: M \to N'$ are A-module hom, for a k-algebra A s.t. $\pi \circ j : N \to M \to N'$ is an iso, then $M = \operatorname{im} j \oplus \ker \pi$.

Pf: Assume $\mathrm{Char}(k) \nmid |G|$, so $|G| \neq 0$ in k. Let $W \subseteq kG$ be a submodule of kG. We'll prove that W has a complement module C with $kG = W \oplus C$.
It would follow that kG is c.r. and hence s.s.

We'll obtain C by consider the sequence $\underset{N}{W} \xrightarrow{j} \underset{M}{kG} \xrightarrow{\pi} \underset{N'}{W}$

where j is the natural inclusion hom. $(w \mapsto w \ \forall w \in W)$ and π is a carefully selected hom

s.t. $\pi \circ j = \mathrm{Id}_W$ (which is certainly an iso.)

We'll take $C := \ker \pi$. The recalled lemma then implies that

$$kG = \mathrm{im}\, j \oplus \ker \pi = W \oplus C.$$

Thus, the main task is to construct the hom π s.t. $\pi \circ j = \mathrm{Id}_W$.

π, via the "averaging trick": First take an arbitrary vec. space complement V of W in kG. Thus, $kG = W \oplus V$.

But we don't know if it's a mod. hom.

Let $p : kG = W \oplus V \to W$ be the projection from kG onto W; it's certainly a linear map.

"f"

and define the map $\pi : kG \to W$, $\quad m \longmapsto \frac{1}{|G|} \sum_{g \in G} g\left(p\left(g^{-1} \cdot m\right)\right)$

"γ"

We claim that π is a kG-module hom and $\pi \circ j = \mathrm{Id}_W$.

(1) (2)

(2): $W \xrightarrow{j} kG \xrightarrow{\pi} W \qquad \forall m \in W$, $g^{-1} m \in W$, so $p(g^{-1} \cdot m) = g^{-1} \cdot m$,

so $g \cdot p(g^{-1}(m)) = g \cdot g^{-1} \cdot m = m$ so

$$\pi \circ j(m) = \pi(m) = \frac{1}{|G|} \sum_{g \in G} m = \frac{1}{|G|} \cdot |G| \cdot m = m.$$

(1) Need: $\pi: kG \to W$, $m \mapsto \frac{1}{|G|} \sum_{g \in G} g\, p(g^{-1} w)$ is a kG-mod. hom.

linearity: routine, essentially follows from the linearity of g^{-1}, p, g. $\forall g \in G$.

π respects kG-action: We need to show that $\forall m \in kG$ and $h \in G$,

$$h . \pi(m) = \pi(h . m)$$ $\to$ key idea: a change of variable. (try it!)

$\downarrow$

next time: the details.

consequences of A.W + Maschke's Thm.