Math 4140. Lecture 34.

04.12.2021.

Last time: · proved the remaining ingredients necessary for A.W. thm.

$$\cdot\ \mathrm{End}_A\left(\bigoplus_{i=1}^{r} U_i\right) \cong \Lambda^{U} := \left\{ [\varphi_{ij}] : \varphi_{ij} \in \mathrm{Hom}_A(U_j, U_i) \right\}$$

missing in EH. $\leftarrow$ *

$$\cdot\ \mathrm{End}\left(\bigoplus_{j=1}^{n_i} S_j^{(i)}\right) \cong M_{n_i}\left(\mathrm{End}_A(S_1^{(i)})\right) \quad \forall i \quad \left(\underset{\text{s.s}}{A} = \bigoplus_{i=1}^{r}\bigoplus_{j=1}^{n_i} \underset{\text{simple ideals}}{S_j^{(i)}}\right)$$

Today: · revisiting the A.W. thm:

- proof outline
- corollaries & uniqueness of A.W. decomposition
- first examples with $A = k[x]/\langle f\rangle$.

1. The Artin-Wedderburn Thm.

Thm. (Thm 5.9. structure of s.s. algebras) Let A be a k-algebra. Then A is s.s. iff there exist $n_1, \dots, n_r \in \mathbb{Z}_{\geq 1}$ and division algebras $D_1, \dots, D_r$ over k s.t.

$$A \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r).$$

Pf: (sketch, again) if: ✓. only if: A s.s $\Rightarrow A = \bigoplus_{i=1}^{r} \bigoplus_{j=1}^{n_i} S_j^{(i)} \Rightarrow \{S_j^{(i)} : i, j\}$

$$A \overset{5.4}{\cong} \operatorname{End}_A(A)^{op} = \operatorname{End}_A\Big(\bigoplus_{i,j} S_j^{(i)}\Big)^{op} \overset{5.6}{\cong} \Lambda^{op} \overset{\substack{\simeq 5.7 \\ \text{Schur's Lemma} \\ + \\ \text{linear algebra}}}{\cong} \Big(\prod_{i=1}^{r} \operatorname{End}_A\big(\bigoplus_j S_j^{(i)}\big)\Big)^{op}$$

$$\overset{\text{our } *}{=} \Big(\prod_i M_{n_i}(\underset{\operatorname{End}_A(S_1^{(i)})}{\widehat{D}_i})\Big)^{op} \overset{\text{H.W.}}{=} \prod_i \Big(M_{n_i}(\widehat{D}_i)^{op}\Big) \overset{\substack{5.8 \\ \text{H.W.}}}{\cong} \prod_i M_{n_i}(\underbrace{\widehat{D}_i^{op}}_{=: D_i}). \quad \square$$

What about uniqueness ?

Prop. (Cor. 5.11. simple modules from A.W. decomps.) Suppose A has decomp. (*). Then

(a) $M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$, hence A, has exactly r simple modules. (1) The simple modules are iso. to $D_i^{n_i}$ (module of column vectors) (2); here $\dim_k(D_i^{n_i}) = n_i \dim_k(D_i)$. (3)

(b) If $k = \bar{k}$ (i.e. k is alg. closed) and A is f.d., then $A \cong M_{n_1}(k) \times \cdots \times M_{n_r}(k)$ (4) and A has precisely r simple modules; these modules are iso to k^{n_i} and hence have dimensions n_i over k. (5)

pf: (sketch). (1) The simples of the direct prod are the inflations of the simples of the components $M_{n_i}(D_i)$ in the sense of Cor. 3.31.

(2) the only simple of $M_{n_i}(D_i)$ is $D_i^{n_i}$ up to iso.

(3) linear algebra. (4) A f.d. $\overset{3.20}{\Rightarrow}$ $S_j^{(i)}$ is f.d. $\overset{\text{Schur}}{\Longrightarrow}$ $D_i = k$. (5) Same as in part (a).

$S_j^{(i)} \cong A/\text{some max. ideal}$.

Fact: The decomposition $A \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$ for a s.s. algebra A is unique up to reordering of the factors, that is, if there is another decomp $A \cong M_{n_1'}(D_1') \times \cdots \times M_{n_s'}(D_s')$, then $r = s$ and there is a permutation $\pi \in S_r$ s.t. $n_i = n'_{\pi(i)}$ and $D_i \cong D'_{\pi(i)}$ $\forall i$.

Rmk: The proof uses things like the number of iso classes of simple A-modules to recover quantities like r and s (see the previous prop.). However, the proof is not trivial or immediate from the A.W. theorem. In particular, one needs to show that $M_n(D) \cong M_{n'}(D') \Rightarrow n = n', D \cong D'$.

Note, If k is algebraically closed, then a f.d. s.s k-algebra A is commutative iff $A \cong k \times k \times \cdots \times k$, i.e., iff all the matrix algebra factors in the A.W. decomp are k. For general k, A s.s, f.d $\Rightarrow n_i = 1 \; \forall i$ in $(*)$.

Pf: (idea). Otherwise A wouldn't be comm since $M_n(k)$ is not comm $\forall n > 1$.

E.g. (1) $k = \mathbb{C} = \bar{k}$. $f = x^2+1 \in k[x]$. $\Rightarrow A = k[x]/\langle f \rangle$ (s.s, f.d., comm.)

$$\underset{\substack{\text{irr} \qquad \text{irr}}}{(x-i)^1 (x+i)^1}$$

$$= \underbrace{k[x]/(x-i)}_{\cong k} \times \underbrace{k[x]/(x+i)}_{\cong k}$$

(2) $k = \mathbb{R} \neq \bar{k}$, $f = \underbrace{x^2+1}_{\text{irr. over } k=\mathbb{R}} \in k[x] \Rightarrow A = k[x]/\langle f \rangle$ (s.s., f.d., comm.)

$$= \mathbb{R}[x]/\langle x^2+1 \rangle \cong \mathbb{C}$$

for more systematic treatment of A.W. decomp of $k[x]/\langle f \rangle$, see § 5.3.

$M_1(D)$, $D = \mathbb{C}$, a division alg. over $k = \mathbb{R}$.