Math 4140. Lecture 33.

04.09.2021.

Last time:

- proved that $A \cong \mathrm{End}_A(A)^{op}$
- introduced the algebra $\Lambda^{(u)} := \left\{ \left[\varphi_{ij} \right] : \varphi_{ij} \in \mathrm{Hom}_A(U_j, U_i) \right\}$
  for A-modules $U_1, \dots, U_r$.
- Constructed the map $\Phi : \mathrm{End}_A(\underbrace{\oplus U_i}_{V}) \longrightarrow \Lambda^{(u)}$
  $$\gamma \longmapsto \left[\varphi_{ij} := \pi_i \circ \gamma \circ \kappa_j \right]$$
  for A-modules $U_1, \dots, U_r$, where κ_j, π_i are the natural incl. and proj. maps

Today:

- Prove that Φ is an algebra iso.
- prove that for each i in $A = \bigoplus_{i=1}^{r} \bigoplus_{j=1}^{n_i} S_j^{(i)}$, A.s.s. $S_j^{(i)}$ simple ideal
  $$\Lambda^{(S)} \cong M_{n_1}(D_1) \times M_{n_2}(D_2) \times \cdots \times M_{n_r}(D_r)$$
  where $D_i = \mathrm{End}_A(S_1^{(i)})$.
- deduce A.W. (with uniqueness)

1. Pf that Φ is an iso.

$$\Phi : \mathrm{End}_A(\underbrace{\oplus U_i}_{V}) \longrightarrow \Lambda = \{[\varphi_{ij}] : \varphi_{ij} \in \mathrm{Hom}_A(U_j, U_i)\}$$

$$\gamma \longmapsto [\gamma] := [\varphi_{ij}^{\gamma} = \underline{\pi_i} \circ \gamma \circ \underline{\kappa_j}].$$

Pf: (sketch)

Recall that $\pi_j \kappa_i \overset{①}{=} \delta_{ij} \cdot \mathrm{Id}_{U_i}$ and $\sum_{i=1}^{r} \kappa_i \pi_i \overset{②}{=} \mathrm{Id}_V$.

- Φ makes sense (the outputs are actually in Λ): ✓ domain ✓, codomain ✓, hom. ✓
- Φ is linear: $\forall \beta, \gamma \in \mathrm{End}_A(V)$, $a, b \in k$,

$$\pi_i \circ (a\beta + b\gamma) \circ \kappa_j = a(\pi_i \beta \kappa_j) + b(\pi_i \gamma \kappa_j) = a\varphi_{ij}^{\beta} + b\varphi_{ij}^{\gamma}. \quad \forall i, j \quad ✓$$

- Φ is unital: $\varphi_{ij}^{\mathrm{Id}_V} = \pi_i \mathrm{Id}_V \kappa_j = \pi_i \kappa_j \overset{①}{=} \delta_{ij} \mathrm{Id}_{U_i} \Rightarrow \Phi(\mathrm{Id}_V) = \mathrm{Id}_\Lambda$.
- Φ is multiplicative: $\forall \beta, \gamma \in \mathrm{End}_A(V)$, we have

$$\Phi(\beta)\,\Phi(\gamma) = \Phi(\beta\gamma) \qquad \text{since} \; \ldots$$

$\left(\Phi(\beta)\Phi(\gamma)=\Phi(\beta\gamma)\right)$: $\quad ([\beta][\gamma])_{ij} = \sum_{k=1}^{r} \beta_{ik}\gamma_{kj} = \sum_{k=1}^{r} \pi_i \beta \kappa_k \pi_k \gamma \kappa_j$

$$= \pi_i\beta\Big(\underbrace{\sum_{k=1}^{r} \kappa_k\pi_k}_{\mathrm{Id}_V\ (3)}\Big)\gamma\kappa_j = \pi_i(\beta\gamma)\kappa_j = [\beta\gamma]_{ij} \qquad \forall\, i,j.$$

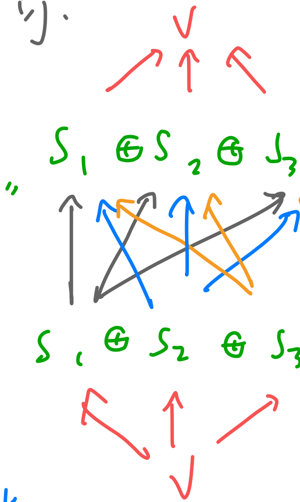
Surj of Φ:

Let $[\varphi_{ij}]\in\Lambda$. Consider

$$\gamma := \sum_{k=1}^{r}\sum_{l=1}^{r} \kappa_l\,\varphi_{lk}\,\pi_k$$

Note: the map $[\varphi_{ij}]\mapsto\gamma$ is actually the inv of Φ.

e.g. "$S = U$"



We claim that $\Phi(\gamma)=[\gamma]=[\varphi_{ij}]$. We need $\gamma_{ij}=\varphi_{ij}\ \forall\, i,j$.

nonzero only when $i=l,\ j=k$

$$\gamma_{ij} = \pi_i\left(\sum_{k=1}^{r}\sum_{l=1}^{r} \kappa_l\,\varphi_{lk}\,\pi_k\right)\kappa_j = \sum_k\sum_l \underbrace{\pi_i\kappa_l}_{\delta_{il}\mathrm{Id}_{U_l}}\varphi_{lk}\underbrace{\pi_k\kappa_j}_{\delta_{kj}\mathrm{Id}_{U_j}} = \varphi_{ij}$$

inj of Φ: Ex.

Our proof is now complete.

Where are we now?

Given s.s. $A = \bigoplus_i \bigoplus_{j=1}^{n_i} \underbrace{S_j^{(i)}}_{\text{simples}}$, we have

$$A = \mathrm{End}_A(A)^{op} = \underline{\mathrm{End}\Big(\bigoplus_{i=1}^{r} \bigoplus_{j=1}^{n_i} S_j^{(i)}\Big)}^{op}$$

$$= \Lambda^{(S)} := \Big\{ \big[\varphi_{(i,j)(k,l)} \big] : \varphi_{(ij)(kl)} \in \underbrace{\mathrm{Hom}_A\big(S_l^{(k)}, S_j^{(i)}\big)}_{\substack{\shortparallel \\ 0 \text{ unless } k=i.}} \Big\}$$

$$= \left\{ \begin{bmatrix} S^{(1)}\text{-block} & 0 & 0 \\ 0 & \cdots & 0 \\ 0 & 0 & S^{(r)}\text{-block} \end{bmatrix} \right.$$

$S_1^{(1)} \oplus S_2^{(1)} \oplus S_1^{(2)}$

$S^{(1)}$-block : $\begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix}$

$\psi_{12} : S_2^{(1)} \to S_1^{(1)}$.

$S^{(i)}$ block : $\begin{bmatrix} \psi_{11} & \psi_{12} & \cdots & \psi_{1n_i} \\ & & \vdots & \\ & & & \psi_{n_in_i} \end{bmatrix}$

$\psi_{ab} : S_b^{(i)} \to S_a^{(i)}$.

Next: show that $\forall i$, $\Big\{ \big[\psi_{ab}^{(i)} \big]_{\substack{a,b \\ \in \\ \{1,\ldots,n_i\}}} \Big\} \cong M_{n_i}(D_i)$

where $D_i := \mathrm{End}_A(S_1^{(i)})$

2. $\Lambda^{(i)} \cong M_n(D_i)$

Note that the "i" is superficial : it suffices to show that for iso. simples $U_1, -, U_n$, $\Lambda^{(u)} \cong M_n(D)$ where $D = \mathrm{End}_A(U_1)$.

$(U_j \leftrightarrow S^{(i)}_j)$

Pf: (main idea) $\forall\, 1 < j \leq n$, pick an module iso $\phi_{j1} : U_1 \to U_j$ and let $\phi_{1j} = \phi_{j1}^{-1}$. Take $\phi_{11} = \mathrm{Id}_{U_1}$. Consider the map

$$\Psi : \Lambda^{(u)} \to M_n(D), \quad [f_{ij}] \longrightarrow \left[g_{ij} := \phi_{1i} f_{ij} \phi_{j1}\right] \in M_n(D).$$

Claim: Ψ is an iso. E.x. $\begin{cases} \text{linearity, inj. surj, unitality: check coordinatewise, easy.} \\ \text{use that } \phi_{ij}\text{'s are isos} \\ \text{multiplicativity: routine, use } \phi_{ij}\phi_{ji} = \mathrm{Id}_{U_i}. \end{cases}$

Consequence:

$$A \cong \begin{pmatrix} \Lambda^{s^{(1)}} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \Lambda^{s^{(r)}} \end{pmatrix}^{op} \cong \begin{pmatrix} M_{n_1}(D_1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & M_{n_r}(D_r) \end{pmatrix}^{op}$$

linear alg.

$$\overset{\downarrow}{\cong} \left(M_{n_1}(D_1) \times \cdots M_{n_r}(D_r) \right)^{op} = M_{n_1}(D_1)^{op} \times \cdots \times M_{n_r}(D^r)^{op}$$

$$\cong \underbrace{M_{n_1}(D_1^{op}) \times \cdots \times}_{\text{A.W. decomp.!}} M_{n_r}(D_r^{op}).$$

careful statement, corollaries, examples: next time!