Math 4140. Lecture 32.

04. 07. 2021.

Last time: reduced the proof of the A.W. thm to

(1) $A \cong \boxed{\mathrm{End}_A(A)^{op}}$ for an algebra A.

(2) $A = \bigoplus_{i=1}^{r} \bigoplus_{j=1}^{n_i} S_j^{(i)}$ s.s., $S_j^{(i)}$ simple w/ $S_j^{(i)} \cong S_{j'}^{(i')} \Leftrightarrow i = i'$.

$\Rightarrow \mathrm{End}_A(A) = \mathrm{End}_A\left(\bigoplus_{i,j} S_j^{(i)}\right) \cong \prod_i \mathrm{End}_A\left(\bigoplus S_j^{(i)}\right) \cong \prod_i M_n(D_i)$

$D_i \cong \mathrm{End}(S_1^{(i)})$

Today: · prove (1); start the proof of (2).

1. $A \cong \mathrm{End}_A(A)^{op}$

Prop. (Lemma 5.4.) Let A be a k-algebra. Then there is an alg. iso

$\psi : A \to \mathrm{End}_A(A)^{op}$, $a \mapsto \Big(r_a : A \to A$, the end. with $r_a(x) = xa \;\; \forall a \in A \Big)$

r_a → right mult. by a

Pf: The map ψ makes sense (each output $\psi(a) = r_a$ is indeed an A-mod hom): fix $a \in A$

Then r_a is easily seen to be linear.

$\forall b \in A, x \in A$, $\quad r_a(b \cdot x) = (b \cdot x) a = bxa = b(xa) = b \cdot r_a(x)$.

$\overset{\shortparallel}{V}$

linearity of ψ: we need to show that $\psi(a+b) = \psi(a) + \psi(b)$ and

$\psi(\lambda a) = \lambda \psi(a)$, i.e., $\underbrace{r_{a+b} = r_a + r_b}_{(1)}$ and $\underbrace{r_{\lambda a} = \lambda r_a}_{(2)}$, $\forall a, b \in A, \lambda \in k$.

$\forall a' \in A$, $\quad r_{a+b}(a') = a'(a+b) = a'a + a'b = r_a(a') + r_b(a') = (r_a + r_b)(a')$

$r_{\lambda a}(a') = a'(\lambda a) = \lambda a'a = \lambda(r_a(a')) = (\lambda r_a)(a')$, so (1) and (2) hold.

injectivity of ψ: suppose $\psi(a) = 0$. i.e., $r_a = 0$. Then $r_a(a') = aa' = 0$ $\forall a' \in A$.

In particular, for $a' = 1$ we have $a = a \cdot 1 = 0$,

so $a = 0$, therefore ψ is inj.

Surjectivity of ψ: take $\varphi \in \mathrm{End}_A(A)^{op} \overset{set}{=} \mathrm{End}_A(A)$, so that $\underset{*}{\underline{\varphi \text{ is an } A\text{-module}}}$

hom. let $a = \varphi(1)$. Then $\forall a' \in A$,

$$\varphi(a') = \varphi(a'\cdot 1) \overset{*}{=} a'\cdot\varphi(1) = a'\cdot a \quad = a'a = r_a(a')$$

so $\varphi = r_a = \psi(a)$. therefore ψ is surj.

ψ is unital: we need $\psi(1) = id_A$, ie., $r_1 = id_A$, ie., $r_1(a') \overset{\star}{=} a' \; \forall a' \in A$.

But $r_1(a') = a'\cdot 1 = a'$. so $\star$ does hold for all $a' \in A$.

ψ respects mult: we need $\psi(ab) = \underbrace{\psi(a) \cdot \psi(b)}_{\text{mult in } \mathrm{End}_A(A)^{op}} = \underbrace{\psi(b)\cdot\psi(a)}_{\text{mult in } \mathrm{End}_A(A)} = \psi(b)\circ\psi(a)$,

ie., that $r_{ab} = r_b \circ r_a \quad \forall a,b \in A$.

Now, $\forall a' \in A$, $r_b \circ r_a(a') = r_b(a'a) = a'ab = a'(ab) \quad = r_{ab}(a')$,

so $r_{ab} = r_b \circ r_a$, as desired. It now follows that ψ is an alg. iso. $\square$

2. from $\mathrm{End}_A(A)^{op}$ to matrices

$$(\mathrm{End}_A(A))^{op} = \mathrm{End}_A\Big(\bigoplus_{i,j} S_j^{(i)}\Big)^{op} = \Big(\prod_{(i)} \mathrm{End}\big(\bigoplus_j S_j^{(i)}\big)\Big)^{op}$$

$$= \Big(\prod_{(i)} M_{n_i}(D_i)\Big)^{op}$$

Step 1. matrices with homomorphisms as entries.

Prop: (Lemma 5.5) Let A be a K-algebra. Let $U_1, \dots, U_r$ be A-modules.

The set

$$\Lambda := \Big\{ \Big[\ \underbrace{\varphi_{ij}}_{\text{entry in Row } i,\ \text{Col } j}\ \Big]_{ij} : \varphi_{ij} \in \mathrm{Hom}_A(U_j, U_i) \Big\}$$

i) a K-algebra under matrix add. and mult, where product of matrix entries are given by composition.

$$([f][g])_{ij} = \sum_{k=1}^{r} f_{ik}\, g_{kj}$$

E.g. $p=3$.

$$\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \cdot \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} = \begin{bmatrix} f_{11}g_{11} + f_{12}g_{21} + f_{13}g_{31} & \cdots & \cdots \\ \cdots & \cdots & f_{21}g_{13} + f_{22}g_{23} + f_{23}g_{33} \\ \cdots & \cdots & \cdots \end{bmatrix}.$$

Pf: (sketch) Λ is a vector space: routine check.

Λ is an algebra: We already argued that Λ is closed under multiplication. The identity elt of Λ is the matrix

$$I = \begin{bmatrix} id_{U_1} & & & 0 \\ & id_{U_2} & & \\ & & \ddots & \\ 0 & & & id_{U_r} \end{bmatrix}.$$

All axioms for algebras are routine to check (Ex. check the id. prop. for I)

$\square$

Step 2. $\mathrm{End}_A(\oplus U_j) \cong \Lambda$

Prop: (Lemma 5.6) Let A be a k-algebra and let $U_1, \dots, U_r$ be A-modules. Then the matrix algebra Λ is iso, as an alg, to $\mathrm{End}_A\left(\bigoplus_{j=1}^{r} U_j\right)$.

pf: (sketch) Construction of the iso: Write $V = \bigoplus_{j=1}^{r} U_j$.

We need the natural inclusion hom. $\kappa_i : U_i \to V$ and projection hom $\pi_i : V \to U_i$ for $1 \le i \le r$. Note that (i) $\pi_i \circ \kappa_i = \mathrm{id}_{U_i}$ $\forall i$ and (ii) $\sum_{i=1}^{r} \kappa_i \circ \pi_i = \mathrm{id}_V$

useful

We'll show that the map

$$\Phi : \mathrm{End}_A(V) = \mathrm{End}_A\left(\bigoplus_{j=1}^{r} U_j\right) \longrightarrow \Lambda, \quad \gamma \longmapsto \left[\gamma_{ij} \right],$$

pf: next time

where $\gamma_{ij} : U_j \to U_i$ is given by $\gamma_{ij} = \pi_i \circ \gamma \circ \kappa_j$ $\forall i,j$, is an iso of algebras.