Math 4140. Lecture 31.

04.05.2021.

Last week: · properties of Jacobson radicals · spring pause lectures

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Part of Ch 4. Semisimple modules/algebras → nice, simple submodules serve as building blocks for s.s. modules.

This week: structure of s.s. algebras : the Artin-Wedderburn Thm. (A.W.)

1. Statement of the A.W. Thm.

Thm. Let K be a field and A a k-algebra. Then A is semisimple iff it is isomorphic to an algebra of the form

(Thm 5.9.)

$$A \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r) \qquad (*)$$

where $n_1, \ldots, n_r \in \mathbb{Z}_{\geq 1}$ and $D_1, \ldots, D_r$ are division algebras over k.

Moreover, when A is s.s. the decomposition is unique up to reordering of the factors.

2. Proof ingredients / strategy

Ingredient (0) the if part / working with division algebras.

Prop: (Lemma 5.8)

(1). For any division algebra D over k, the matrix alg. $M_n(D)$ is s.s.

(2). An algebra of the form $M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$ is s.s.

Pf: (1) $\Rightarrow$ (2) since direct prod. of s.s. algebras are s.s.

(1): Same proof as for s.s. of $M_n(k)$, can show

$$M_n(D) = \bigoplus_{i=1}^{n} C_i \;\longrightarrow\; \left\{ \left[\begin{array}{cccc|c|ccc} 0 & 0 & \cdots & 0 & d_1 & 0 & \cdots & 0 \\ & & & & d_2 & & & \\ & & & & \vdots & & & \\ & & & & d_n & & & \end{array}\right] : d_1, \ldots, d_n \in D \right\}.$$

where each C_i is iso to D^n and simple.

Pf: E.X.

So we have proved the "if" direction.

Now suppose that A is a s.s. algebra with decomp

$$A = \left(S_1^{(1)} \oplus S_2^{(1)} \oplus \cdots \oplus S_{n_1}^{(1)}\right) \oplus \left(S_1^{(2)} \oplus S_2^{(2)} \oplus \cdots \oplus S_{n_2}^{(2)}\right) \oplus \cdots \oplus \left(S_1^{(r)} \oplus S_2^{(r)} \oplus \cdots S_{n_r}^{(r)}\right)$$

into simple ideals where $S_j^{(i)} \cong S_{j'}^{(i')}$ iff $i = i'$. (So $S_j^{(i)} \cong$ some simple mod. $S^{(i)}$ $\forall\, 1 \le j \le n_i$ for each i)

Note that the multiplicities $n_1, \cdots, n_r$ are uniquely determined because n_j is just the multiplicity of the comp factor $S^{(i)}$ in a comp. series of A.

Ingredient (1). An important algebra is $\underline{B \cong \mathrm{End}_B(B)^{op}}$ (works for any algebra B).
Lemma 5.4.

What's $\mathrm{End}_B(B)^{op}$? • View B as the regular module $V = B$. The notation $\mathrm{End}_B(B)$ is just $\mathrm{End}_B(V)$, which means the end. algebra of A-mod. end. of V as usual: $\mathrm{End}_B(B) = \{ f : \underset{V}{B} \to \underset{V}{B} \mid f \text{ is } B\text{-module hom} \}$.

• For any alg. C (here $C = \mathrm{End}_B(B)$), C^{op} is the algebra with the $\begin{cases} \text{same underlying v.s.} \\ \text{opposite multiplication.} \end{cases}$

"opposite multiplication" : $\forall a,b \in C^{op}$, $a \underset{C^{op}}{\cdot} b = b \underset{C}{\cdot} a$. **Ex:** The alg. axioms hold for C^{op}.

Pf: We establish an iso $B \longrightarrow \mathrm{End}_B(B)^{op}$ for any K-algebra B. (Lemma 5.4)

<u>The map</u>:

<u>linearity</u>:

<u>Injectivity</u>:

<u>Surjectivity</u>:

<u>hom. property</u>:

skipped for now, will prove next time.

$\square$

Ingredient 2. Schur's Lemma.

e.g. say $A = S_1^{(1)} \oplus S_2^{(1)} \oplus S_1^{(2)}$

$S_1^{(1)} \oplus S_2^{(1)} \oplus S_1^{(2)}$

$S_1^{(1)} \oplus S_2^{(1)} \oplus S_1^{(2)}$

Thm (Schur's Lemma, Thm 3.33) Let S be a simple module of a k-algebra A.

- The algebra $\tilde{D} = \mathrm{End}_A(S)$ is a division algebra.
- If k is alg. closed and $\dim(S) < \infty$, then $\mathrm{End}_A(S) = k$.
- If T is another simple A-module with $S \not\cong T$, then $\mathrm{Hom}_A(S, T) = \{0\}$.

non-iso simples "don't talk to each other".

What will happen: break $A \cong \mathrm{End}_A(A)^{op} \cong \left[\mathrm{End}_A\left(\bigoplus_{i,j} S_j^{(i)}\right)\right]^{op}$

into $\left[\mathrm{End}_A\left(\bigoplus_j S_j^{(1)}\right) \times \cdots \times \mathrm{End}_A\left(\bigoplus_j S_j^{(r)}\right)\right]^{op} \overset{(ii)}{\cong} \left[M_{n_1}(\tilde{D}_1) \times \cdots \times M_{n_r}(\tilde{D}_r)\right]^{op}$

(i) $\cong$

To make this precise we need (i) and (ii) Lemma 5.5, Lemma 5.6 & Thm 5.7. $\to$ Wednesday

Ingredient 3. cleaning up / removing the "op". Lemma 5.8. (a).

Lemma: For any division algebra $\tilde{D}$ and $n \in \mathbb{Z}_{\geq 1}$. we have

(1) $D := \tilde{D}^{op}$ is a division algebra. ✓

(2). $[M_n(\tilde{D})]^{op} \cong M_n(\tilde{D}^{op})$.

Application: $$A \cong [M_{n_1}(\tilde{D}_1) \times \cdots \times M_{n_r}(\tilde{D}_r)]^{op} \overset{\text{E.x.}}{=} M_{n_1}(\tilde{D}_1)^{op} \times \cdots \times M_{n_r}(\tilde{D}_r)^{op}$$

$$\cong M_{n_1}(\tilde{D}_1^{op}) \times \cdots \times M_{n_r}(\tilde{D}_r^{op})$$

$$\cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$$

where $D_i = \tilde{D}_i^{op}$ and hence a div. alg. $\forall i$. $\square$

finishes the proof of the A.W. Thm.

Pf of the lemma: (sketch)

(1) We already know that $\tilde{D}^{op}$ is an algebra, so it suffices to show that every nonzero elt $a \in \tilde{D}^{op}$ has an inverse. The inverse $b = a^{-1}$ in A will do, since $a \underset{\tilde{D}^{op}}{*} b = b \underset{\tilde{D}}{\cdot} a = 1$ and similarly $b \underset{\tilde{D}^{op}}{*} a = a \underset{\tilde{D}}{\cdot} b = 1$.

(2) The two algebras are identical as sets: they are both

$$\{n \times n \text{ matrices w/ entries from } \hat{D} \overset{\text{set.}}{=} D\}.$$

EX: Show this gives an iso. Use $(XY)^T = Y^T X^T$.

An iso from $M_n(\tilde{D}^{op})$ to $[M_n(\tilde{D})]^{op}$ can be given by the transpose map $X \mapsto X^T_D$.

It remains to investigate Ingredient 1 (Lemma 5.4) and

Ingredient 2 (Lemmas 5.5, 5.6, Thm 5.7) carefully. → next time!