Math 4140. Lecture 28.

03.29.2021.

Last time: · almost finished the proof of Thm 4.23, on

Today: 1. finish the proof 2. applications of the theorem.

1. It remains to prove that $\Phi: A/J \longrightarrow A/M_1 \oplus \cdots \oplus A/M_r$, $a+J \longmapsto (a+M_1, \ldots, a+M_r)$ is an A-module iso where A is a finite-length alg. $J=J(A)$, and $M_1, \ldots, M_r$ a minimal list of maximal submodules of A s.t. $J=\bigcap_{i=1}^{r} M_i$.

It further suffices to show that $E_i := (0, \ldots, \underbrace{1+M_i}_{i\text{th spot}}, 0, \ldots, 0) \in \operatorname{Im}\Phi$, for these E_i's generat the module $\bigoplus_{i=1}^{r} A/M_i$.

Consider the module $\bigcap_{j\neq i} M_j$. By assumption, $\bigcap_{j\neq i} M_j \not\subseteq M_i$. Thus, since

M_i is a maximal submodule of A, we have $M_i + \bigcap_{j\neq i} M_j = A$.

In particular, we have $1_A = m_i + y$ for some $m_i \in M_i$, $y \in \bigcap_{j\neq i} M_j$.

It follows that

$$\begin{aligned}\Phi(y+J) &= (y+M_1, y+M_2, \dots, y+M_i, \dots, y+M_r) \\ &= (0, 0, \dots, \underline{1-m_i+M_i}, \dots, 0) \\ &= (0, 0, \dots, \underline{1+M_i}, \dots, 0) \\ &= E_i.\end{aligned}$$

We are done. $\square$

2. Applications of Thm 4.23.

→ E.x. HW 8.(7) / E.g. 4.22.(2).

(a). Semisimplicity criterion for algebras of the form $A = k[x]/\langle f\rangle$.

$f = f_1^{a_1} \cdots f_r^{a_r}$

Strategy: · A is f.d., so A has finite length and Thm 4.23 applies. We'll compute the Jacobson radical J of A and determine when A is s.s by determining when $J = 0$. We find J by computing the intersection of all max. left ideals of A.

· to find the intersection of the max. ideals, use the Correspondence Thm to find these ideals, then use the fact that in $k[x]/\langle f\rangle$, $\bigcap_{i=1}^{r} \langle g_i\rangle/\langle f\rangle = \langle g_1 g_2 \cdots g_r\rangle/\langle f\rangle$ if $g_1, \ldots, g_r$ are pairwise coprime.

$\left(p \in \langle g_i\rangle\ \forall i \Rightarrow g_i \mid p\ \forall i \Rightarrow \underline{\mathrm{Lcm}}(g_1, \ldots, g_r) \mid p \Rightarrow p \in \langle \underline{\mathrm{Lcm}}\rangle \right)$

E.g. $f = (x-1)^2(x-2)$. ⇝ max. ideal. $\langle g\rangle/\langle f\rangle$, g irr. $g \mid f$.

$\langle x-1\rangle/\langle f\rangle$, $\langle x-2\rangle/\langle f\rangle$

(b). Semisimplicity criterion for path algebras of acyclic quivers.

Prop. Let Q be an acyclic quiver and let $A = kQ$. Then the Jacobson radical J of A is the subspace of A spanned by all paths of positive length in Q.

Note: For any algebra A, isomorphic modules M, N have equal annihilators, i.e., $\mathrm{Ann}_A(M) = \mathrm{Ann}_A(N)$ by preservation of scalar actions.

pf: Recall (from Lecture 18. Mar. 01.) that the modules $S_i = Ae_i / J_i$, $1 \le i \le r$ is a complete set of simple modules of A up to isomorphism, so $J = \bigcap_{i=1}^{r} \mathrm{Ann}_A(S_i)$.

Here $r = |Q_0|$. $J_i = Ae_i^{\geq 1}$ is the span of all paths in Q starting at vertex i that have positive length.

we assume $Q_0 = \{1, 2, \cdots, r\}$

Note that $\mathrm{Ann}_A(S_i) = \mathrm{Ann}_A\left({}^{Ae_i}/_{J_i} \right) \underset{\swarrow}{=} J_i \oplus \left(\bigoplus_{j \neq i} Ae_j \right)$

E.x. immediate from the def. of the A-action. $p \cdot \left({}^{qe_i}/_{J_i} \right) = pqe_i + J_i$.

It follows that $J = \bigcap_{i=1}^{r} \underbrace{J_i \oplus \left(\bigoplus_{j \neq i} Ae_j \right)}_{\text{span of all paths except } e_i}$ = the ideal equalling the span of all positive-length paths in Q. $\square$

Corollary: (Corollary 4.27) Let $A = kQ$ for an acyclic quiver Q. Then A is s.s. iff Q has no arrows. Moreover, if Q has no arrows, then we have

$$A \overset{*}{=} \underbrace{k \times k \times \cdots \times k}_{r \text{ times},\ r = |Q_0|} \quad \text{as an algebra.}$$

Pf: The first claim follows immediately from the prop. A desired iso. for the second statement can be given by $A = kQ = \operatorname{Span}\{e_1, \cdots, e_r\} \to k \times k \times \cdots \times k$,

$$e_i \longmapsto (0, \cdots, 0, \underset{i\text{th spot}}{1}, 0, \cdots) \ldots\ldots$$

Rmk: (1). The decomposition $k \times k \cdots \times k$ of the s.s. algebra A in the second statement is a special case of the Artin-Wedderburn Thm: any s.s. algebra is of the form $M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$ for some division algebras $D_1, \cdots, D_r$ over k.

(2). decomposition into simples vs. indecomposables. When Q has any arrow, kQ

is not s.s so decomposing $A = kQ$ into simple submodules is not that

"interesting". It will still be interesting to decompose A into indecomposables.

another natural family of "building blocks" for arbitrary modules. → Ch. 7. Krull-Schmidt Thm.

(3). Recall that

$A = kQ$ is not s.s

$\Updownarrow$

some A-module V is not s.s

$\Updownarrow$

some A-module V is not completely reducible → Can we find such an A-module?

(i.e., V has a proper, nontrivial submodule w/ no complement)

Next time: Artin-Wedderburn Thm.