Math 4140. Lecture 27.

03. 26. 2021.

Last time: · Def. & Properties of Jacobson radicals.

intersection of all maximal left ideals

Thm. (Thm 4.23.) Let A be a k-alg. of finite length. Let J be its Jacobson radical.

(a). J equals to union of finitely many max. left ideals.

(b). We have $J = \bigcap_{S \text{ simple}} \mathrm{Ann}_A(S)$.

(c). J is a two-sided ideal.

✓

(d). J is nilpotent, and $J^n = 0$ where n is the length of A.

(d') J contains every nilpotent left ideal of A.

J is the largest nilp. left ideal.

Pf of (d) and (d') : (we'll restate and prove (e) – (h) later.

(d). "J is nilpotent." (Strategy : hit the comp. factors of A with J.)

Take a composition series $0 = V_0 \subset V_1 \subset V_2 \cdots \subset V_n = A$ of A as an A-module.

Then V_i / V_{i-1} is simple for all $1 \le i \le n$, so by (b) $J \cdot \left(V_i / V_{i-1} \right) = 0$, i.e., $J \cdot V_i \subseteq V_{i-1}$.

$(j \cdot (a + V_{i-1}) = ja + V_{i-1})$

It follows that $J^n \cdot \underset{\overset{\cup}{1}}{A} = J^n \cdot V_n = V_0 = 0$, so $J^n = 0$.

(d'). Lemma: $\forall x \in A$, we have $x \in J \Leftrightarrow 1 - rx$ has a left inverse for every $r \in A$.

b s.t. $b(1-rx) = 1$.

Pf: "$\Rightarrow$": Let $\begin{matrix} x \in J \\ r \in A \end{matrix}$. We prove that $A(1-rx) = A \ni 1$. If not, then $A(1-rx)$ is a proper ideal of A. hence by Zorn's Lemma $1-rx$ is contained in a maximal ideal M.

Thus, $x \in J \subset M$, so $rx \in M$, so $1 = (1-rx) + rx \in M$. But then $A \subseteq M$, a contradiction.

"$\Leftarrow$" We prove the contrapositive: if $x \notin J$, then $\exists$ max. ideal M s.t. $x \notin M$, so $M + Ax = A$. In particular, $1 = y + rx$ for some $y \in M$ and $r \in A$. But then $1 - rx = y$ has no left inverse: if it did, then $\exists z$ s.t. $zy = 1 \in M$, which again would imply $A \subseteq M$, a contradiction.

E.x. Use the lemma and the formula

$$1 - (rx)^m = (1 + rx + (rx)^2 + \cdots + (rx)^{m-1})(1 - rx)$$

to prove (d').

the theorem, continued:

(e). A/J is s.s

(f) If $I \subseteq A$ is a two-sided ideal s.t. A/I is s.s, then $J \subseteq I$.

} J is smallest two-sided ideal K s.t. A/K is s.s.

(g) A is s.s $\iff$ $J = 0$ — J measures s.s. of A.

(h) An A-module V is s.s. $\iff$ $J \cdot V = 0$. — J measures s.s. of A-module.

Pf: (e). We will show that A/J is a s.s. A-module. Since J annihilates A/J, it would follow from Thm 4.17 that A/J is also a s.s. A/J-module. Thus by def, A/J is s.s.

(Lec25. use $A=A$, $I=J$, $B=A/J$, $V=A/J$)

To show that A/J is a s.s. A-module, we show that there is an iso

$$\Phi: A/J \longrightarrow A/M_1 \oplus A/M_2 \oplus \cdots \oplus A/M_r$$ of A-modules where $M_1, \ldots, M_r$ are

$$a+J \longmapsto (a+M_1, a+M_2, \ldots, a+M_r)$$

max. left ideals of A s.t.

$$J = \bigcap_{i=1}^{r} M_r$$

and. $\bigcap_{j\neq i} M_j \not\subset M_i$.

Since A/M_i is simple (as an A-module) $\forall\, 1\le i\le r$, it follows that A/J is s.s. as an A-module

Pf that Φ gives us an iso. as claimed:

(i). Φ is well-defined: $a+J=b+J \Rightarrow a-b\in J\subseteq M_i \Rightarrow a+M_i=b+M_i\ \forall a,b\in A, 1\le i\le r. \Rightarrow \Phi(a+J) = \Phi(b+J)$.

(ii). Φ is indeed an A-module hom: "the action is just mult. on the coset reps on both sides". EX. $\Phi(r.(a+J)) = r.\Phi(a+J)$.

(iii). Φ is inj: $a+J\in\ker\Phi \Rightarrow a+M_i=0\in A/M_i\ \forall i \Rightarrow a\in M_i\ \forall i \Rightarrow a\in \bigcap_{i=1}^{r} M_i = J \Downarrow a+J=0$.

(iv). Φ is surj: It suffices to show that $(0, 0, \ldots, 1+M_i, 0, \ldots, 0)$ is $\operatorname{Im}\Phi\ \forall i$.

Postponed.

(f). Suppose A/I is s.s., then we have $A/I = S_1 \oplus \cdots \oplus S_r$

as A/I-modules where $S_1, \ldots, S_r$ are simple A/I-modules

By the proof of Corollary 4.12, each S_i is also a simple A-module,

so $J \cdot S_i = 0 \ \forall i$. But then $\underbrace{J(A/I)}_{=0} = J \cdot (S_1 \oplus \cdots \oplus S_r) = 0$. ie,

$J = JA \subset I$. $\square$

(g). If A is s.s, then $A = A/0$ is s.s., so $J \leq 0$ by (f), so $J = 0$.

If $J = 0$, then $A = A/J$ is s.s. by (e).

(f). " $\Rightarrow$ " V is a s.s. A-module $\Rightarrow V = \bigoplus_{i=1}^{r} S_i$, S_i a simple A module $\forall i$.

$\Rightarrow J \cdot S_i = 0 \ \forall i$, so $J \cdot V = J \cdot (\bigoplus S_i) = 0$

" $\Leftarrow$ ": $J \cdot V = 0 \xRightarrow{\text{Lemma 2.37.}} V$ is an A/J-module

$\Rightarrow V$ is a s.s. A/J module since A/J is s.s

$\xRightarrow{\text{Thm 4.17.}} V$ is a s.s. A-module.

- pf that Φ from (e) is surj.
- Examples / Applications of Jacobson radicals

— next time!