Math 4140. Lecture 26.

03. 24. 2021.

Last time:

- A k-algebra A is s.s $\Longleftrightarrow$ every A-module is s.s.
- Subalgebras of s.s. algebras may not be s.s. but hom. images, iso. copies, quotients, direct sums and direct summands of s.s. algebras are s.s.

needs proof

Today:

- finish the proof
- Jacobson radical (Thm 4.23)

Prop. (direct sum/summands) Let $A_1, \dots, A_r$ be finitely many k-algebras. Then the direct product $A := A_1 \times A_2 \times \cdots \times A_r$ is a s.s. algebra iff each A_i is a s.s. algebra.

Recall our preparation

(a) Let $\varepsilon_i = (0, 0, \dots, \underset{i\text{th spot}}{1_{A_i}}, 0, \dots, 0)$. Then for every A-module M, we have

$$M \overset{*}{=} \bigoplus_{i=1} \varepsilon_i M$$

as A-modules. (Lemma 3.30)

$A_i \cong A / \mathrm{Span}\{\varepsilon_j : j \neq i\}$

(b) Consider the natural projection $\pi_i : A \to A_i$, $(a_1, a_2, \dots, a_r) \mapsto a_i$

It is a surj hom and has kernel $\ker \pi_i = \mathrm{Span}\{\varepsilon_j : j \neq i\}$.

(c) Given an algebra A and a quotient algebra $B = A/I$, a B-module V is s.s as a B-module iff it's a s.s. A-module with $IV = 0$. (Thm 4.17.)

Pf: "only if": If A is s.s, then for each $1 \le i \le r$ the algebra A_i is s.s. because A_i is iso. to a quotient of A by (b).

"if": Suppose $A_1, A_2, \cdots, A_r$ are all s.s. We show that A is s.s. by proving that every A-module M is s.s.

Consider the decomp. $M \overset{*}{=} \bigoplus_{i=1}^{r} \varepsilon_i M$. It suffices to show that $M_i := \varepsilon_i M$ is s.s. as an A-module. Note that M_i is an A_i-module via the action. $A_i \times M_i \to M_i$,

$$\underset{\substack{\cap \\ A_i}}{a_i} \cdot \big((0,0,\cdots 1_i, 0 \cdots) \cdot \underset{\substack{\cap \\ M}}{m} \big) = \underbrace{(0, \cdots, a_i, 0, \cdots 0)}_{\substack{\text{ith spot} \\ \cap \\ A}} \cdot m.$$

But A_i is s.s., so M_i is a s.s. A_i-module. Also note that in $A \curvearrowright M_i$, we have $\varepsilon_j \cdot M_i = 0$ $\forall j \neq i$. Thus,

$$\left\{ \begin{array}{l} M_i \text{ is a s.s. } A\text{-module, hence a s.s. } A/\mathrm{Span}\{\varepsilon_j \mid j \neq i\} \text{ module} \\ \varepsilon_j \cdot M_i = 0 \quad \forall j \neq i \end{array} \right. \xRightarrow{\text{Thm 4.17}} M_i \text{ is a s.s. } A\text{-module.}$$

$\square$

Jacobson Radical

Let A be a k-algebra.

Def: The Jacobson radical $J(A)$ of A is the intersection of all maximal left ideals of A.

Note: It's not called a "left Jacobson ideal"... The reason is it's a two-sided ideal and also the intersection of all maximal right ideals. [see Schiffler §4.1)

Def (nilpotent ideals) (Recall that given two left ideals $I, J \subseteq A$, the set $IJ = \mathrm{Span}_k\{xy \mid x \in I, y \in J\}$ is again a left ideal; in particular, we have a chain of ideals $I \supseteq I^2 \supseteq I^3 \supseteq I^4 \supseteq \cdots$)

We say an ideal $I \subseteq A$ is nilpotent if the ideal I^r is zero for some $r \geq 1$.

Def: (annihilator of modules) Let M be an A-module. The <u>annihilator of M</u> is the set
$$\mathrm{Ann}_A(M) = \{ a \in A \mid a \cdot m = 0 \ \forall m \in M \}.$$

Ex. (HW) For any A module M, the set $\mathrm{Ann}_A(M)$ is a two-sided ideal of A.

Thm 4.23. Suppose that <u>A has finite length.</u> Let $J = J(A)$. Then the following holds.
ie, A has a comp series

(a) J is the intersection of finitely many maximal left ideal.

(b) We have
$$J = \bigcap_{S \text{ simples}} \mathrm{Ann}_A(S).$$

(c) J is a two-sided ideal.

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(d). J is a nilpotent ideal. Moreover, if n is the length of J, then $J^n = 0$.

J is the largest nilp. left ideal.

(d'). J contains every nilpotent left ideal I of A. → not in the book, but in Schiffler

(e). A/J is s.s.

(f). If $I \subseteq A$ is any two-sided ideal s.t. A/I is s.s, then $J \subseteq I$.

J is the smallest 2-sided ideal whose corr. quotient is s.s.

(g). A is s.s. iff $J = 0$. — J measures s.s. of the algebra A.

(h). An A-module V is s.s. iff $J \cdot V = 0$ — J can detect s.s. of A-modules.

We'll start the proof today ...

Pf: (a). (sketch) "finite intersection" If not, we can find an infinite chain of ideals $(A \supseteq) M_1 \supsetneq M_1 \cap M_2 \supsetneq M_1 \cap M_2 \cap M_3 \supsetneq M_1 \cap M_2 \cap M_3 \cap M_4 \supsetneq \cdots\cdots$ of intersections of maximal left ideals where all containments are proper. Write $I_j = M_1 \cap M_2 \cap \cdots \cap M_j$. then we have I_j / I_{j-1} is simple for all $j > 1$ (Ex. use the 2nd Iso Thm). This cannot happen since A has finite length.

(b) & (c): (b). $J = \bigcap_{S \text{ simple}} \underbrace{\mathrm{Ann}_A(S)}_{\text{two-sided ideal by Ex.}}$, so (b) $\Rightarrow$ (c). (c) J is a two-sided ideal.

We prove (b) next.

" $J = \bigcap_{S \text{ simple}} \mathrm{Ann}(S)$ " : We first prove that $J \supseteq \bigcap_{S \text{ simple}} \mathrm{Ann}(S)$. Suppose $a \in A$ is an elt which annihilates every simple module S of A; we'll show that $a \in M$ for every maximal left ideal and hence in J. Consider the quotient module $\underset{S_M :=}{A/M}$ of the regular module. Since M is maximal, S_M is simple. Thus, a annihilates A/M. It follows that $a \cdot A \subseteq M$ and thus $a \in M$ $\left(a \cdot S_M = 0 \Rightarrow \underset{\forall a' \in A}{a \cdot (a' + M) = aa' + M} = 0 \Rightarrow \underset{\forall a' \in A}{a a' \in M} \Rightarrow \underset{a'' = 1}{a \cdot 1} \in M \right)$.

Next we prove that $J \subseteq \bigcap_{S \text{ simple}} \mathrm{Ann}(S)$ by proving that $J \subseteq \mathrm{Ann}(S)$, ie, $JS = 0$, for an arbitrary simple A-module S. Note that $J \cdot S$ is a submodule of S since J is a left ideal. If $JS = 0$, then we are done. If not, we must have $JS = S$ since S is simple. But then for each nonzero $s' \in S$, we have $J \cdot s' = S$; in particular, $\exists j \in J$ s.t. $j \cdot s' = s'$! But then $(j-1) \cdot s' = 0$, hence $j - 1 \in \mathrm{Ann}(s')$. But $j \in J \subseteq \underset{A}{\overset{I}{\overset{\shortparallel}{\mathrm{Ann}(s')}}}$ $\left(\begin{array}{l} A/I \cong As' = S, \\ \text{so } I \text{ is maximal} \end{array} \right)$ so $1 = j - (j-1) \in \mathrm{Ann}(s')$. This cannot happen since $1 \cdot s' = s' \neq 0$.

more proofs next time...