Math 4140. Lecture 25.

03.22.2021.

Last time:

1. equivalent definitions of s.s. modules:

   complete reducible, "being a sum of simples"

2. closure properties: hom. images, iso copies, quotient, submodule

   direct sums, direct summands of s.s. modules are s.s.

Today: Semisimple algebras:

· examples

Main Thm. → · An algebra A is s.s. iff every A-module is s.s.

(i.e., the regular A-mod is s.s.)

· Closure properties: do we still have all the closure properties from 2?

1. Key examples:

(a) We've already seen that any matrix algebra $M_n(k)$ is s.s.

We'll see today that direct products of such algebras are also s.s.

(Later: Artin-Wedderburn Thm says that every s.s. alg. is $\cong$ to such a direct prod.)

(b) We saw that $T_n(k)$ is not s.s.: nonzero submodules of $V = k^n$ intersect nontrivially, so V is not a direct sum of them, so $T_n(k)$ cannot be s.s. by the main theorem.

Note: (a) and (b) show that a subalgebra of a s.s. algebra may not be s.s.

For discussion on when an algebra of the form $k[x]/\langle f\rangle$ is s.s., see Prop 4.14.

2. All modules of a s.s. algebra are s.s.

Thm 1. (Thm 4.11.) Let A be a k-algebra. Then A is s.s. iff all modules of A are s.s.

Pf: The "if" implication holds trivially. To prove the "only if", suppose A is a s.s algebra and let V be an A-module. Take a basis $\beta = \{v_i \mid i \in I\}$ of V. Consider the map $\psi: \bigoplus_{i\in I} A \to V$, $(a_i)_{i \in I} \longmapsto \sum_{i\in I} a_i v_i$.

It is easy to see that ψ is an A-module hom: (Ex)

(e.g. additive: $\psi((a_i)+(b_i)) = \psi((a_i+b_i)) = \sum (a_i+b_i)v_i = \sum a_iv_i + b_iv_i = \sum a_iv_i + \sum b_iv_i = \psi((a_i)) + \psi((b_i))$.

scalar: ... respects A-action: $\psi(a\cdot(a_i)) = \psi((aa_i)) = \sum (aa_i)v_i = \sum a(a_iv_i)$

$= a\cdot \sum a_iv_i = a\psi((a_i))$)

$(\psi : \bigoplus_{i \in I} A_i \to V)$ Now, ψ is surjective because $V_i \in \mathrm{Im}\,\psi$ $\forall i \in I$, so $V = \mathrm{Im}\,\psi$.

Since A is a s.s. module by assumption, $\bigoplus_{i \in I} A_i$ is s.s. It follows that the hom. image $\mathrm{Im}\,\psi = V$ is s.s. $\square$

3. New s.s. algebras from old.

We saw from the example $T_n(k) \leq M_n(k)$ that a subalg. of a s.s. algebra A may not be s.s. What about algebras related to A in other ways?

Prop. (homomorphic images) Let $\varphi: A \to B$ be a surjective algebra hom. If A is s.s, then B is s.s.

Recall: Using an alg. hom $\varphi: A \to B$, we may induce an A-module from any B-module V by the action. $a \cdot v = \varphi(a) \cdot v \quad \forall a \in A, v \in V.$

Pf: We'll show B is s.s by showing that every B-module V is s.s. Induce V to an A-module. Then since A is s.s, as an A-module we have $M \overset{A}{=} \sum_{i \in I} S_i$ where S_i is simple $\forall i \in I$. We hope to show that each S_i here is automatically a B-module and a simple B-module, so that $M \overset{B}{=} \sum S_i$ and M is s.s, as desired.

S_i is a B-module: We show S_i is a submodule of V as a B-module because: (1). S_i is a subspace of V since S_i is an A-submodule of V.

(2). S_i is closed under the B-action, i.e., $b.s \in S_i$ $\forall s \in S_i$:

φ is surj $\Rightarrow b = \varphi(a)$ for some $a \in A$ $\Rightarrow$ $b \cdot s = \varphi(a) \cdot s = a \cdot s \underset{=}{\in} S_i$. ✓

S_i is an A-module.

S_i is a simple B-module: Let U be a nonzero B-submodule of S_i. Then it again may be viewed (by "induction") as an A-submodule of S_i. Since S_i is a simple A-module, it follows that $U = S_i$. We just proved that a B-submodule of S_i is either zero or S_i, therefore S_i is a simple B-submodule.

Corollary 1: (Iso copies) If A and B are iso. algebras, then A is s.s iff B is s.s.

Pf: Take an iso $\varphi: A \to B$, consider the surj hom φ and φ^{-1}, and apply the prop. $\square$

Corollary 2: (Quotients) Let A be a s.s. algebra and let I be a two-sided ideal of A. Then A/I is s.s.

Pf: Apply the proposition to the natural projection hom $\pi: A \to A/I$, $a \mapsto a+I$ which is surj. $\square$

Thm 2. (Thm 4.17.) Let A be a k-algebra and let I be a two-sided proper ideal of A. Let $B = A/I$ and view every B-module V an A-module via the induced action $a \cdot v = \pi(a) \cdot v = (a+I) \cdot v \quad \forall a \in A, v \in V$.

Then for every B-module V, we have

$$V \text{ is a s.s. } B\text{-module} \iff V \text{ is a s.s. } A\text{-module such that } I \cdot V = 0.$$

"going between modules of an algebra (A) and its quotients"

Pf: E.x. see [EH].

Corollary 3. (direct sum/summands) Let $A_1, \cdots, A_r$ be finitely many k-algebras. Then the direct product $A := A_1 \times A_2 \times \cdots \times A_r$ is a s.s. algebra iff each A_i is a s.s. algebra.

Preparation: (a) Let $\varepsilon_i = (0, 0, \cdots, \underset{i\text{th spot}}{1_{A_i}}, 0, \cdots, 0)$. Note that for every A-module M, we have $M = \bigoplus_{i=1}^{r} \varepsilon_i M$ as A-modules. (Lemma 3.30)

(b) Consider the natural projection $\pi_i : A \to A_i$, $(a_1, a_2, \cdots, a_r) \mapsto a_i$. It is a surj hom and has kernel $\ker \pi_i = \mathrm{Span}\{\varepsilon_j : j \neq i\}$.

Pf: next time!