Math 4140. Lecture 24. 03.17.2021.

Last time: Overview of semisimple modules and s.s. algebras.
("s.s.": $\oplus$ of simples) (includes all $\mathbb{C}G$, G a finite gp)

Today: · semisimple modules: examples. definitions and properties.
(mostly "closure properties")

· "A is a s.s. algebra $\Longleftrightarrow$ all modules of A are s.s."
($A \curvearrowright A$ is s.s.)

1. Examples of s.s./non-s.s modules

(0). Simple modules are s.s.

(1). $A = M_n(k) \overset{\text{regular}}{\circlearrowright} V = M_n(k)$

$$V = M_n(k) = \bigoplus_{i=1}^{n} C_i \cong \bigoplus_{i=1}^{n} \underbrace{k^n}_{\text{simple}} \quad \text{is s.s.}$$

(2) $A = T_n(k) \circlearrowright V = k^n$. There are exactly $n+1$ submodules
upper Δ.

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V = k^n$$

where $V_i = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ 0 \\ \vdots \\ 0 \end{bmatrix} : x_1, x_2, \ldots, x_i \in k \right\}$ by Ex. 2.14. In particular, no two nonzero submodules of V have intersection, so V cannot be the direct sum of a set of nonzero submodule, so V is not s.s.

2. Definition of s.s. modules.

Thm 4.3. Let A be a k-alg and let V be a nonzero A-module. TFAE:

(1) (def.) V is s.s. i.e., V is a direct sum of simple submodules.

(2) (complete reducibility) For every submodule U of V, there is a submodule C of V s.t. $V = U \oplus \underline{C}$.

"complement of U"

useful characterizations of s.s.

(3) (sum) V is a sum of simple submodules, i.e., $V = \sum_{i \in I} S_i$ for some index set I, where S_i is simple for all $i \in I$

Remarks on the proof: (1) implies (3) since direct sums are sums. It then suffices to prove that (3) implies (2) and that (2) $\Rightarrow$ (1). These two proofs need Zorn's Lemma and we'll skip it for now.

3. Properties of s.s. modules.

Let A be a k-alg.

The alternative characterizations of s.s. modules imply the following:

Corollary 1. (submodules $\equiv$ quotients) Let V be a s.s. A-module. Then every submodule of V is iso to a quotient module of V, and vice versa.

Pf: Use the complete reducibility condition / characterization:

"sub. $\Rightarrow$ quotient": Let U be a submodule. Then it has a complement C s.t. $V = U \oplus C$. But then $V/C = {}^{U\oplus C}/_{C} = {}^{U+C}/_{C} = {}^{U}/_{U\cap C} = {}^{U}/_{0} \cong U$.

"quotient $\Rightarrow$ sub.": Let W be a quotient of V. say $W = V/U'$. Then U' has a complement C' s.t. $V = U' \oplus C'$. Thus, $W = V/U' = {}^{U'\oplus C'}/_{U'} = {}^{U'+C'}/_{U'} \cong {}^{C'}/_{U'\cap C'} = C'$. $\square$

Lemma: Let $\varphi: S \to V$ be an A-module hom, with S simple.

Then either $\varphi = 0$ or $\operatorname{Im}\varphi$ is a simple A-submodule iso. to S.

$\downarrow$ In other words, homomorphic images of simples are either zero or simple

Pf sketch:

$$S \text{ is simple} \Rightarrow \text{dichotomy} \begin{cases} \varphi = 0 \\ \varphi \neq 0 \Rightarrow \ker\varphi \neq S \Rightarrow \ker\varphi = 0 \Rightarrow \operatorname{Im}\varphi \cong S/\ker\varphi \cong S. \end{cases}$$

Corollary 2. (Homomorphic images of s.s. modules are s.s.)

(Cor. 4.7.) Let $\varphi: V \to W$ be a nonzero A-mod. hom. If V is s.s. then $\operatorname{Im}\varphi$ is s.s.

In particular, if φ is surjective ($\operatorname{Im}\varphi = W$) then W is s.s.

Pf: (sketch) V is s.s. $\Rightarrow$ $V = \sum_{i \in I} S_i$, simples $\Rightarrow$ $\operatorname{Im}\varphi = \varphi(V) = \varphi\left(\sum_{i\in I} S_i\right) = \sum_{i \in I} \underbrace{\varphi(S_i)}_{\text{simple or zero by the Lemma}} \xRightarrow[\text{the zeros}]{\text{ignore}}$ $\operatorname{Im}\varphi$ equals a sum of simples $\Rightarrow$ V is s.s

Corollary 3. (Isos. preserve s.s.) Two isomorphic A-modules are either both s.s or both non-s.s. i.e., if $\varphi: V \to W$ is an iso of A-modules, then V is s.s. iff W is s.s.

pf: If V is s.s, then $W = \operatorname{Im}\varphi$ is s.s. by Corollary 2.

If W is s.s., then we consider $\varphi^{-1}: W \to V$ (automatically an iso):

by Corollary 2, $V = \operatorname{Im}\varphi^{-1}$ is s.s.

Corollary 4. (Submodules/quotients of s.s. are s.s.) Let V be a s.s. A-module. Then every submodule of V is s.s, and every quotient module of V is s.s.

Pf sketch: Use

- "quotients are the same as homomorphic images" HW. 7.(5).

  V/W. W a submodule $\rightsquigarrow$ $\pi: V \to V/W$, $v \mapsto v+W$ $\Rightarrow$ $V/W \cong \operatorname{Im}\pi$.

- "Homomorphic images of s.s. modules are s.s." + "Iso. preserves s.s"

  $\longrightarrow$ $V/W \cong \operatorname{Im}\pi$ is s.s.

- "Submodules and quotients of s.s. modules are the same" + "Iso preserves s.s" $\longrightarrow$ U a submodule of V $\Rightarrow$ $U \cong V/C$ for the complement ... ...

Corollary 5. (Direct sum / direct summands of s.s. modules are s.s.)

Let $(V_i)_{i\in I}$ be a family of nonzero A-modules. Then $\bigoplus_{i\in I} V_i$ is s.s iff V_i is s.s $\forall i\in I$

pf: Recall that is a natural inj. inclusion map $\iota_i : V_i \to V := \bigoplus_{i\in I} V_i$ with $\operatorname{Im}\iota_i \cong V_i$. If V is s.s. then $\operatorname{Im}\iota_i$, being a submodule of V, is s.s. so V_i is s.s. $\forall i$. Conversely, if V_i is s.s. then $V_i = \bigoplus_{j\in J_i} S_{ij}$, so

$$V = \bigoplus V_i = \sum_{i\in I}\iota_i(V_i) = \sum_{i\in I}\iota_i\Big(\bigoplus_{j\in J_i} S_{ij}\Big) = \sum_{i\in I,\, j\in J_i} \underbrace{\iota(S_{ij})}_{\text{either simple or zero}} \Rightarrow V \text{ is s.s.} \quad \square$$

Summary: Homomorphic images, isomorphic copies, submodules, quotient modules, direct sums, and direct summands of s.s. modules are s.s.

Next time: s.s. algebras.