Math 4140. Lecture 23.

Midterm corrections: due 11:59 pm Mar. 24th.

03.15.2021.

Last time:

- Consequences of Schur's Lemma

$A \circlearrowright V$ f.d. simple:

- $a \in Z(A) \Longrightarrow a\cdot = \lambda \, \mathrm{Id}_V$ for some $\lambda \in k$.
- A is commutative $\Longrightarrow \dim V = 1$.

- Simple modules of $k[x]/\langle f \rangle$:

If f has unique factorization $f = f_1^{a_1} \cdots f_r^{a_r}$ into distinct irreducibles $f_1, \dots, f_r$, then $k[x]/\langle f_1 \rangle, k[x]/\langle f_2 \rangle, \dots, k[x]/\langle f_r \rangle$ give a complete, irredundant list of non-iso. simples of $k[x]/\langle f \rangle$.

Today: · Overview of Ch 4 ('b): Semisimple modules and semisimple algebras.

Let A be a k-algebra.

Def 1: (semisimple modules) A semisimple (S.S.) A-module is a nonzero A-module V which equals the direct sum of simple submodules. ($V = \bigoplus_{i \in I} S_i$)

E.g. (1) Simples are semisimple. (2). $A = k \Rightarrow$ All A-modules are S.S.: $V = \bigoplus_{i \in I} S_i$

All A-modules $\cong$ k-v.s. ; V: take basis $\{v_1, v_2, \dots\}_I$, then take $S_i = \mathrm{Span}(v_i)$

(3). $A = M_n(k) \curvearrowright V = M_n(k) = \bigoplus_{i=1}^{n} C_i \Rightarrow V$ is a S.S. A-module.

$M_n(k) \curvearrowright k^n$; C_i: column modules iso to k^n

Def 2: (semisimple algebras) We call A a semisimple (S.S.) algebra if the left regular module $V = A$ ($A \curvearrowright A$) is a semisimple module.

E.g. By Ex. (3) above, $A = M_n(k)$ is semisimple.

... (2) , k is semisimple ($k = M_1(k)$).

Remarkable Facts:

(1). **Thm 4.11**: Let A be a k-algebra. Then A is semisimple if and only if every nonzero module of A is semisimple! (The "if" is clear by definitions, but the "only if" also holds.)

(2). Every semisimple algebra is a direct product of matrix algebras (up to iso):

Thm 5.9: Suppose A is s.s. Then there exist $r, n_1, \ldots, n_r \in \mathbb{Z}_{\geq 1}$ and division k-algebra $D_1, D_2, \ldots, D_r$ s.t.

$$A = M_{n_1}(D_1) \times \underbrace{M_{n_2}(D_2) \times \cdots \times M_{n_r}(D_r)}_{B}. \quad (*)$$

Conversely, an algebra of the form B is semisimple.

If k is alg. closed, then A is s.s. $\Rightarrow$ $A \cong M_{n_1}(k) \times M_{n_2}(k) \times \cdots \times M_{n_r}(k)$ for some $r, n_1, \ldots, n_r \in \mathbb{Z}_{\geq 1}$.

"Artin-Wedderburn decomp"

13) Group algebras of finite gps are usually semisimple:

Thm 6.3 (Maschke's Thm) Let k be a field and G a finite gp. Then kG is s.s. iff the characteristic of k does not divide $|G|$.

(In particular, if $\mathrm{char}(k) = 0$ then kG is s.s.)

e.g. $\mathbb{R}, \mathbb{C}, \mathbb{Q}$

Thm 6.4. We can explicitly describe the Artin-Wedderburn decomposition of kG when kG is s.s. In particular, if $k = \mathbb{C}$, the number of factors r in $(*)$ is the number of iso classes of simples of kG, and $n_1, n_2, \dots, n_r$ are the dimensions of the simples.

finite!

E.g. $G = S_3$. $k = \mathbb{C}$. $A = kG \circlearrowright V^3 = \langle v_1, v_2, v_3 \rangle$

Maschke : $\mathbb{C}S_3$ is s.s.

Artin-Wedderburn: $\overbrace{\mathbb{C}S_3}^{6d} = \overbrace{M_1(\mathbb{C})}^{1d} \times \overbrace{M_2(\mathbb{C})}^{4d} \times ?$

On the other hand, by Midterm: $\mathbb{C}S_3$ has simples $U = \mathrm{Span}\{v_1+v_2+v_3\}$ 1d

$W = \{av_1 + bv_2 + cv_3 : a+b+c \overset{!}{=} 0\}$ 2d

Consequence: There is at most / precisely one more factor in "?" in the AW decomposition. Correspondingly,

kG has a 1-d simple non-iso to U or W.

Q: What is this third simple? ("the sign rep": $kG \to M_1(\mathbb{C}) \cong \mathbb{C}$, $g \mapsto \mathrm{sgn}(g)$)

Next time: S.S. modules.