Math 4140. Lecture 22.

03.12.2021.

Last time: · We can essentially understand representations of an arbitrary f.d. k-algebra via representations of some quiver (which satisfy certain relations).

$\mathrm{Hom}_A(S,T)=0$ if $S\not\cong T$.
There's no ~~nontrivial~~ trivial homs between distinct (non-iso.) simples.

· **Schur's Lemma:** Let A be a k-alg and let S,T be simple modules of A.

(a). Every A-module hom $\phi: S\to T$ is either 0 or an iso. In particular, $\mathrm{End}_A(S)$ is a division algebra.

(b). If S is f.d and k is alg. closed, then every $\phi\in\mathrm{End}_A(S)$ equals $\phi=\lambda\,\mathrm{id}_S$ for some $\lambda\in k$.

Note: (1) Last time I stated the first part of A as "Every A-module hom $\phi: S\to S$ (1) is either zero or an iso". Today's is the more general version I should have given.

(2) For (b), we need the fact that $\left\{\begin{array}{l} S \text{ is f.d} \\ k \text{ is alg closed}\end{array}\right\} \Longrightarrow$ every $\phi\in\mathrm{End}_k(S)$ has an e-vector.

Pf of Schur's Lemma:

(a) The same proof as last time (for the special case "$S = T = V$") works:
Say $\phi: S \to T$ is a nonzero A-mod hom. Then $\ker\phi \neq S$ and $\mathrm{Im}\,\phi \neq 0$. Since S, T are simple, it follows that $\ker\phi = 0$ and $\mathrm{Im}\,\phi = T$, so ϕ is inj and surj, hence an iso.

(b) Let $\phi: S \to S$ be any A-mod hom. Then ϕ has an eigenvector v, say with eigenvalue λ. This means that $\phi(v) = \lambda \cdot v$, or $(\phi - \lambda\, id)(v) = 0$. This implies that $\phi - \lambda\, id$ has a nontrivial kernel. By the proof (a), this can happen only if $\phi - \lambda\, id = 0$, so $\phi = \lambda\, id$. $\square$

Applications of Schur's Lemma:

Observation: Actions of central elts on an A-module gives an A-mod hom.

Let A be a k-alg. Let $a \in \underline{Z(A)} = \{z \in A : zb = bz \ \forall b \in A\}$, and let V
"center"
be an A-module. Then the map $Act_a : V \to V$, $v \mapsto a \cdot v$ is a module

hom: linearity. ✓ respects the A-action: $\forall v \in V$, $\forall b \in A$

$$b \cdot (Act_a(v)) \overset{?}{=} Act_a(b \cdot v)$$

$$b \cdot (a \cdot v) \overset{?}{=} a \cdot (b \cdot v)$$

$$(ba) \cdot v \overset{?}{=} (ab) \cdot v$$ Yes, since $ab = ba$.

Corollary. (Lemma 3.37.) Let k be alg. closed, A an k-alg, V a f.d. simple module.

Let $a \in Z(A)$. Then a acts as a fixed scalar on V, i.e., $a \cdot v = \lambda v \ \forall v \in V$ for some $\lambda \in k$.

Pf: By the observation, the a-action gives a hom in $\mathrm{End}_A(V)$. The result then follows from Schur's Lemma (b).

Corollary. (Corollary 3.38.) Let k, A, V be as in the above coro.

If A is commutative, then V is one dimensional. **Pf:** E.x.

$Z(A) = A$

"f.d. simples of comm alg / $k = \bar{k}$ must be 1d."

For the rest of today, we'll discuss simple modules of $k[x]/\langle f \rangle$.

This finishes Ch. 1–3. almost completely (except 3.4.3. simples of direct products of algebras; see Coro. 3.31). We'll start Ch4. next week.

1. Simple modules of $k[x]/(f)$

(EH 3.4.1) Let f be a poly in $k[x]$ of positive degree.

Recall: · $k[x]$ is an Euclidean domain (integral domain with an Eucledean alg.) and hence a PID (principal ideal domain) and hence UFD (unique factorization domain: every nonzero elt can be factored into "irreducible" elts in a unique way).

· This makes $k[x]$ similar to $\mathbb{Z}$ in many senses. (see next page)

Dummit & Foote.

In particular, "primes" = "irreducibles" make up any poly. just as prime numbers make up any integer. (e.g. $\mathbb{Z}$: $20 = 2\times 2\times 5$,

$$x^3+5x^2+x+5 = \begin{cases} (x^2+1)(x+5) & \text{if } k=\mathbb{R} \\ (x-i)(x+i)(x+5) & \text{if } k=\mathbb{C} \end{cases}$$

). Also, for two principal ideals $\langle f\rangle, \langle g\rangle$ in $k[x]$, $\langle f\rangle \subseteq \langle g\rangle \iff g\mid f$ (similar to the fact that in $\mathbb{Z}$ $\langle a\rangle \subseteq \langle b\rangle \iff b\mid a$). Consequently, $\langle f\rangle$ is a maximal ideal in $k[x]$ iff f is prime/irreducible (similar to $\mathbb{Z}$ ...).

Q: What are the simple modules of $A = k[x]/\langle f\rangle$?

Recall: Simple modules are always iso to quotients of regular modules:

$A \curvearrowright V$. If V is simple, then $\forall v\in V, v\neq 0$, we have $V = Av \cong A/\mathrm{Ann}(v)$

where the iso. comes from the A-mod. hom $A \to Av$, $a \mapsto a.v$.

Thus, a simple $\underbrace{k[x]/\langle f\rangle}_{A}$ module must be iso to a simple quotient module of $k[x]/\langle f\rangle$. What are they?

- To get (simple) quotients of A we need (maximal) submodules of A. By the Correspondence Thm, (max.) submodules of $k[x]/\langle f\rangle$ are of the form $B = I/\langle f\rangle$ where I is an (max.) ideal containing $\langle f\rangle$. By the last page, this means B has the form $B = \langle h\rangle/\langle f\rangle$ where $h \mid f$, and h needs to be an irreducible poly. if we want A/B to be simple (eq. B to be maximal).

- It follows that all simple modules of A are of the form $A/B = \dfrac{k[x]/\langle f\rangle}{\langle h\rangle/\langle f\rangle} \cong k[x]/\langle h\rangle$ where h is an irr. poly dividing f.

3rd Iso Thm.

Thm. (Prop 3.23) Let $A = k[x]/\langle f \rangle$ where $f \in k[x]$ is a poly. of positive deg.

(a) The simple A-modules are, up to iso, precisely the A-modules where h is an irr. poly dividing f.

(b) Suppose the unique factorization of f is $f = f_1^{a_1} \cdots f_r^{a_r}$ where $\{f_1, \dots, f_r\}$ are pairwise coprime. Then A has precisely r simples up to iso, namely $k[x]/\langle f_1 \rangle, \dots, k[x]/\langle f_r \rangle$.

Pf: By the last page, it really just suffices to prove that

$$\underbrace{k[x]/\langle f_i \rangle}_{S_i} \not\cong \underbrace{k[x]/\langle f_j \rangle}_{S_j}$$

for different $i, j \in [n]$ now. To do so we use "preservation of scalar actions" again. Recall that for any $h \mid f$, we have $A = k[x]/\langle f \rangle$ action on $k[x]/\langle h \rangle$ by $(p + \langle f \rangle).(p' + \langle h \rangle) = pp' + \langle h \rangle$.

Thus, for $a = f_i + \langle f \rangle \in A$, we have
$$\begin{cases} a.\ g + \langle f_i \rangle = f_i g + \langle f_i \rangle = 0 \quad \forall\, g \in k[x]. \\ a.\ (1 + \langle f_j \rangle) = f_i + \langle f_j \rangle \neq 0 \text{ since } f_i, f_j \text{ are coprime.} \end{cases}$$

that is, $a = 0 \curvearrowright S_i$, $a \neq 0 \curvearrowright S_j$. Done! $\square$