Math 4140. Lecture 21.

03.10. 2021.

Last time:

- Construction of the maps $\left\{\begin{matrix}\text{reps of}\\ Q\end{matrix}\right\} \underset{G}{\overset{F}{\rightleftarrows}} \left\{\begin{matrix}\text{reps of}\\ kQ\end{matrix}\right\}$.
- discussion of the "equivalence."

Today:

- More on F & G. categorical equivalence.
- Schur's Lemma.

1. A categorical equivalence

Let Q be a quiver.

· **Def:** (a) (subreps, simplicity, homomorphisms) Given a rep $V = (V_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ of Q, a <u>subrepresentation</u> of V is a tuple $W = (W_i, \psi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ s.t. · W_i is a subspace of V_i for all $i \in Q_0$, · ψ_α is just the restriction $\varphi_\alpha|_{W_i}$ of φ_α to W_i for every arrow $\alpha : i \to j$ in Q, and · $\psi_\alpha(W_i) \subseteq W_j$ for all such arrows $\alpha : i \to j$.

$$\left(\begin{array}{ccc} \varphi_\alpha : V_i & \longrightarrow & V_j \\ \downarrow \quad \cup\!| & & \cup\!| \\ \varphi_\alpha|_{W_i} : W_i & \longrightarrow & W_j \end{array}\right.$$

; clearly the conditions make W a rep of Q in its own right.)

(b) We say V is simple if $V \underline{\neq 0}$ and the only subrepresentations of V is $\underline{0}$ and V.

$0 = \left(\underset{\forall i}{V_i = 0}, \underset{\forall \alpha}{\varphi_\alpha = 0}\right)$

(c). Let $V' = (V_i', \varphi_\alpha')$ be a rep of Q. A homomorphism from V to V' is a set of linear transformations $\Phi = (\phi_i : V_i \to V_i')_{i \in V}$ that is compatible with the repn. maps in the sense that the following square commutes, ie,

$$\alpha : i \to j \quad \rightsquigarrow \quad \Phi \qquad \begin{tikzcd} V: & V_i \arrow[r, "\varphi_\alpha"] \arrow[d, "\phi_i"'] \arrow[dr, phantom, "\circ"] & V_j \arrow[d, "\phi_j"] \\ V': & V_i' \arrow[r, "\varphi_\alpha'"'] & V_j' \end{tikzcd}$$

$$\phi_j \circ \varphi_\alpha(v) = \varphi_\alpha' \circ \phi_i(v) \quad \forall v \in V_i.$$

An isomorphism of reps is a bijective hom: we say $\Phi = (\phi_i)$ is an isomorphism if each ϕ_i is a bijective hom.

Fact: These def for reps of Q correspond well with the analogs for kQ-modules:

e.g. $W \underset{\text{sub rep for } Q}{\subset} V \rightsquigarrow F(W)$ is a submodule of $F(V)$; a kQ-module M is simple iff GQ is simple; F can take a hom Φ to a module hom.

Upshot: Really, there's an equivalence of categories between

the category of reps of Q (containing the reps and rep homomorphisms) $\longleftrightarrow$ the category of modules of kQ (containing the modules of kQ and module homs)

So we can talk about kQ modules via quiver repns (which requires little more than basic linear algebra).

Bounded quiver algebras

The equivalence $Q\text{-reps} \cong kQ\text{-mod}$ can be generalized to account for relations on a quiver:

Def: A relation on a quiver Q is an elt $r = \sum_{p \in S} c_p \cdot p \in kQ$ where all paths in S share the same source and share the same target.

e.g.

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[r, "\beta"] \arrow[dr, "\delta"'] & 3 \arrow[d, "\gamma"] \\ & & 4 \end{tikzcd}$$

$r = \alpha$ is a relation $\xrightarrow{\text{natural}} \varphi(r) = \varphi_\alpha$

$r' = 3\gamma\beta - 5\delta$ is a relation $\to \varphi(r') = 3\varphi_\gamma \varphi_\beta - 5\varphi_\delta$

Def: A rep (V_i, φ_α) of Q is said to satisfy a relation r if "$\varphi(r)$" is the zero map.

Thm: There is a natural categorical equivalence, for every ideal $I \subseteq kQ$,

$\Downarrow$

Schiffler. 5.2.

$$\left(\begin{array}{c} \text{cat. of reps of } Q \\ \text{satisfying all relation} \\ \text{in } I \end{array} \right) \leftrightarrow \left(\begin{array}{c} \text{cat. of modules} \\ \text{of } kQ/I \end{array} \right)$$

the categories of modules of A and
kQ/I are "equivalent".

Thm: Every finite dim. k-algebra is "Morita equivalent" to a quotient kQ/I for some quiver Q and some ideal $I \subseteq kQ$.

Thus, using quiver representations (linear algebra!) we can essentially understand repns of all finite dimensional k-algebras.

2. Schur's Lemma

k is algebraically closed: every polynomial in $k[x]$ of deg n has n roots in k.

Thm 3.33. (Schur's Lemma) Let A be a k-alg. Let S, T be simple A-modules.

(a) A module hom $\varphi: S \to T$ is either 0 or an isomorphism (hence invertible).

In particular, $\mathrm{End}_A(S) = \{\varphi: S \to S \mid \varphi$ is an A-module hom$\}$ is a division algebra.

(b) If V is f.d. and k algebraically closed, then $\varphi = \lambda\, \mathrm{id}_V$ for some $\lambda \in k$.

Pf: (a). Say $\varphi \neq 0$. Then $\underbrace{\ker\varphi \neq V \text{ and } \mathrm{Im}\,\varphi \neq 0}_{\text{submodules of } V}$. Simplicity of V then forces

$\ker V = 0$ and $\mathrm{Im}\,\varphi = V$, so φ is inj and surj, hence an iso.

(b). Need fact: If k is alg. closed then every k-linear map $f: W \to W$ on a f.d. k-v.s. W has at least one eigenvector. Pf: next time / try it!