

Last time: · Simple modules of path algebras of acyclic quivers.

$$\text{bij: } i \in Q_0 \longmapsto S_i = Ae_i / Ae_i^{\geq 1}$$

· Representations of quivers. Q -reps $\begin{matrix} \xrightarrow{F} \\ \xleftarrow{G} \end{matrix} kQ$ -modules

\downarrow
 $(V_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$

Thm: F and G are mutual inverses.

Today:

- example of the constructions F and G .
- discussion of the theorem
- more facts about F & G . "functoriality".

The theorem, copied from the last lecture.

(1). (G -reps \xrightarrow{F} kQ -modules) Given a rep $(V_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ of Q , we can construct a unique kQ module V s.t. $V = \bigoplus_{i \in Q_0} V_i$ and the linear action is such that $\forall x \in Q^{\pm 1}, i \in Q_0, v_i \in V_i$, $x \cdot v_i = \begin{cases} e_i \cdot v_i = v_i & \text{if } x = e_i \\ e_j \cdot v_i = 0 & \text{if } x = e_j, j \neq i \\ \varphi_\alpha(v_i) \in V_{i'} & \text{if } x = \alpha: i \rightarrow i' \end{cases}$

(2). (kQ -modules \xrightarrow{G} Q -reps) Given a module V of kQ , we can define a rep $(V_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ of Q by setting $V_i = e_i V = \{e_i \cdot v \mid v \in V\}$ $\forall i \in Q_0$ and $\varphi_\alpha: V_i \rightarrow V_{j'}$ to be the map with $\varphi_\alpha(e_i \cdot v) = \alpha \cdot e_i \cdot v = \alpha \cdot v$ $\forall v_i \in V_i$ for each arrow $\alpha: i \rightarrow j'$.
 $(e_j \cdot \alpha) \cdot v \in e_j V = V_{j'}$ \rightarrow ends up in the right space.

(3) The constructions in (1) and (2) are inverse to each other.

In particular, reps of Q are in bijection with modules of kQ .

Useful fact: For any quiver $Q = (Q_0, Q_1)$, the path algebra kQ is generated

by the set $Q^{\leq 1} := \{e_i : i \in Q_0\} \cup Q_1$, subject only to the relations

$$e_i e_j = \delta_{ij} e_i \quad \forall i, j \in Q_0 \quad \text{and} \quad \alpha e_i = \alpha = e_j \alpha \quad \text{for each } \alpha: i \rightarrow j \text{ in } Q_1$$

\Downarrow should imply $\beta e_i = 0$ universal property

lurking here ...

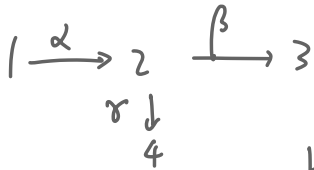
eg:



$$\beta e_1 = \beta e_2 e_1 = \beta \circ = 0$$



eg:



We can construct a kQ -rep $\rho: kQ \rightarrow \text{End}(V)$

by specifying $\rho(e_1), \rho(e_2), \rho(e_3), \rho(e_4), \rho(\alpha), \rho(\beta), \rho(\gamma)$

such that $\rho(e_i) \rho(e_j) \stackrel{(1)}{=} \delta_{ij} \rho(e_i)$ and

$$(2) \left\{ \begin{array}{l} \rho(e_2) \rho(\alpha) = \rho(\alpha) = \rho(\alpha) \rho(e_1), \\ \rho(e_3) \rho(\beta) = \rho(\beta) = \rho(\beta) \rho(e_2), \quad \rho(e_4) \rho(\gamma) = \rho(\gamma) = \rho(\gamma) \rho(e_2) \end{array} \right.$$

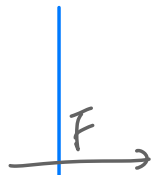
Examples of the constructions F & G .

(a) \mathbb{Q}^α

\mathbb{Q} rep

$(V = V_1, \varphi = \varphi_\alpha)$

$(e_i \cdot V = 1 \cdot V = V, \varphi_\alpha = \varphi)$



same data, one
space + one map

The constructions are obviously inverse to each other.

$$x \cdot v_i = \begin{cases} e_i \cdot v_i = v_i \\ e_j \cdot v_i = 0 \\ \varphi_\alpha(v_i) \in V_i \end{cases}$$

$k\mathbb{Q}$ -rep

$k\mathbb{Q} = k\langle x \rangle$

V

$e_i \cdot v = v \quad \forall v \in V$
 $\alpha \cdot v = \varphi_\alpha(v)$

V , a $k\mathbb{Q}$ -module, so α
acts on V as a map

$\varphi \in \text{End}(V)$

$$(b) \quad 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

$$\quad \quad \quad \gamma \downarrow$$

$$\quad \quad \quad 4$$

Q-reps

e.g.

$$k \xrightarrow{R} k^2 \xrightarrow{S} k^3$$

$$\quad \quad \quad \downarrow T$$

$$\quad \quad \quad R$$

$$e_1 V \xrightarrow{\alpha} e_2 V \xrightarrow{\beta} e_3 V$$

$$\quad \quad \quad \downarrow \gamma$$

$$\quad \quad \quad e_4 V$$

$\rho(\alpha)$
" "
 α



kQ-reps.

$$V = V_1 \oplus V_2 \oplus V_3 \oplus V_4$$

$$= k \oplus k^2 \oplus k^3 \oplus k$$

$$e_2 \cdot (c, \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} u \\ v \\ w \end{bmatrix}, d) = (0, \begin{bmatrix} x \\ y \end{bmatrix}, 0, 0)$$

$$\gamma \cdot (c, \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} u \\ v \\ w \end{bmatrix}, d) = T(\begin{bmatrix} x \\ y \end{bmatrix})$$

$e_i \in V$ corresponding to a rep
 $\alpha, \beta, \gamma \in V$
 $\rho: kQ \rightarrow \text{End}(V)$

Discussion about the proof of Thm 1: (See EH 2.5.2. and "Quiver representations" by Ralf Schiffler, Sections 1.1 & 5.2)

(3) Straightforward once we follow the definitions.

(2) : Since we have proved that $V_i = e_i V$ is a subspace of V and φ_α takes

V_i to V_j for every $\alpha: i \rightarrow j$, it remains to show that φ_α is linear $\forall \alpha \in Q_1$.

This is routine.

$$x \cdot v_i = \begin{cases} e_i \cdot v_i = v_i \\ e_j \cdot v_i = 0 \\ \varphi_\alpha(v_i) \in V_j \end{cases}$$

(1) : Method 1. check that the assignments specified satisfies the necessary relations

Method 2. Define the kQ action on $V = \bigoplus V_i$ more

generally for every path n in kQ : for a path $p = \alpha_k \dots \alpha_1$, from i to j ,

define $p \cdot (v_1, v_2, \dots, v_i, \dots, v_n) = (0, 0, \dots, \underbrace{\varphi_{\alpha_k} \circ \dots \circ \varphi_{\alpha_1}}_{\substack{\text{spot } \hat{V}_i \\ \text{spot } \hat{V}_j}}, 0, \dots, 0)$,

then check this definition satisfies all the module axioms.

See EH. 2.5.2.

□