Math 4140. Lecture 20.

03.05.2021.

Last time:

- simple modules of path algebras of acyclic quivers.

  bij: $i \in Q_0 \longmapsto S_i = Ae_i / Ae_i^{\geq 1}$

- Representations of quivers.

  $\downarrow$

  $(V_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$

  $Q\text{-reps} \underset{G}{\overset{F}{\rightleftarrows}} kQ\text{-modules}$

  Thm: F and G are mutual inverses.

Today:

- example of the constructions F and G.
- discussion of the theorem
- more facts about F & G. "functoriality".

The theorem, copied from the last lecture.

(1). (Q-reps $\xrightarrow{F}$ kQ-modules) Given a rep $(V_i, \varphi_\alpha)_{i\in Q_0, \alpha\in Q_1}$ of Q, we can construct a unique kQ module V s.t. $V = \bigoplus_{i\in Q_0} V_i$ and the linear action is such that $\forall x \in Q^{\geq 1}$, $i\in Q_0$, $v_i \in V_i$,

$$x\cdot v_i = \begin{cases} e_i\cdot v_i = v_i & \text{if } x = e_i \\ e_j\cdot v_i = 0 & \text{if } x = e_j,\ j\neq i. \\ \varphi_\alpha(v_i) \in V_{i'} & \text{if } x = \alpha : i \to i'. \end{cases}$$

(2). (kQ-modules $\xrightarrow{G}$ Q-reps) Given a module V of kQ, we can define a rep $(V_i, \varphi_\alpha)_{i\in Q_0, \alpha\in Q_1}$ of Q by setting $V_i = e_i V = \{e_i\cdot v \mid v\in V\}$ $\forall i \in Q_0$ and $\varphi_\alpha : V_i \to V_j$ to be the map with $\varphi_\alpha(e_i\cdot v) = \alpha\cdot e_i\cdot v = \underset{\substack{\parallel \\ (e_j\cdot\alpha)\cdot v \,\in\, e_j V = V_j}}{\alpha}\cdot v$ $\forall v_i \in V_i$ for each arrow $\alpha: i\to j$.

$\to$ ends up in the right space.

(3) The constructions in (1) and (2) are inverses to each other.

In particular, reps of Q are in bijection with modules of kQ.

Useful fact: For any quiver $Q=(Q_0, Q_1)$, the path algebra kQ is generated by the set $Q^{\leq 1} := \{e_i : i \in Q_0\} \sqcup Q_1$ subject only to the relations

$$e_i e_j \overset{(1)}{=} \delta_{ij} e_i \quad \forall i,j \in Q_0 \quad \text{and} \quad \alpha e_i \overset{(2)}{=} \alpha \overset{(2)}{=} e_j \alpha \quad \text{for each } \alpha : i \to j \text{ in } Q_1$$

⇓ should imply $\beta e_1 = 0$

universal property lurking here ...

e.g.: $\overset{1}{\circ} \xrightarrow{\alpha} \overset{2}{\circ} \xrightarrow{\beta} \overset{3}{\circ}$

$$\beta e_1 = \beta e_2 e_1 = \beta 0 = 0$$

Eg.

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[r, "\beta"] \arrow[d, "\gamma"'] & 3 \\ & 4 & \end{tikzcd}$$

We can construct a kQ-rep $\rho : kQ \to \mathrm{End}(V)$ by specifying $\rho(e_1), \rho(e_2), \rho(e_3), \rho(e_4), \rho(\alpha), \rho(\beta), \rho(\gamma)$ such that

$$\rho(e_i)\rho(e_j) \overset{(1)}{=} \delta_{ij}\, \rho(e_i) \quad \text{and}$$

$$(2) \begin{cases} \rho(e_2)\rho(\alpha) = \rho(\alpha) = \rho(\alpha)\rho(e_1), \\ \rho(e_3)\rho(\beta) = \rho(\beta) = \rho(\beta)\rho(e_2), \quad \rho(e_4)\rho(\gamma) = \rho(\gamma) = \rho(\gamma)\rho(e_2) \end{cases}$$

$$x \cdot v_i = \begin{cases} e_i \cdot v_i = v_i \\ e_j \cdot v_i = 0 \\ \varphi_\alpha(v_i) \in V_{i'} \end{cases}$$

Examples of the constructions F & G.

(a) 

Q rep	kQ-rep $\quad kQ = k\langle x \rangle$.
$(V = V_1,\ \varphi = \varphi_\alpha)$ $\xrightarrow{F}$	V $\quad e_1 \cdot v = v \quad \forall v \in V$, $\quad \alpha \cdot v = \varphi_\alpha(v)$
$(e_1 \cdot V = 1 \cdot V = V,\ \varphi_\alpha = \varphi)$ $\xleftarrow{G}$	V, a kQ-module, so α acts on V as a map $\varphi \in \mathrm{End}\, V)$

same data, one space + one map

The constructions are obviously inverse to each other.

(b)
$$\begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[r, "\beta"] \arrow[d, "\gamma"'] & 3 \\ & 4 & \end{tikzcd}$$

Q - reps $\xrightarrow{F}$ kQ - reps.
$\xleftarrow{G}$

e.g.
$$\begin{tikzcd} k \arrow[r, "R"] & k^2 \arrow[r, "S"] \arrow[d, "T"] & k^3 \\ & k & \end{tikzcd}$$

$$V = V_1 \oplus V_2 \oplus V_3 \oplus V_4$$
$$= k \oplus k^2 \oplus k^3 \oplus k$$

$$e_2 \cdot \left(c, \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} u \\ v \\ w \end{bmatrix}, d \right) = \left(0, \begin{bmatrix} x \\ y \end{bmatrix}, 0, 0\right)$$

$$\gamma \cdot \left(c, \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} u \\ v \\ w \end{bmatrix}, d \right) = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

$$\begin{tikzcd} & \rho(\alpha) \\ e_1V \arrow[r, "\alpha\cdot"] & e_2V \arrow[r, "\beta\cdot"] \arrow[d, "\gamma\cdot"] & e_3V \\ & e_4V & \end{tikzcd}$$

$e_i \circlearrowright V$ corresponding to a rep
$\alpha, \beta, \gamma \circlearrowright$

$\rho: kQ \to \operatorname{End}(V)$

Discussion about the proof of Thm 1:

(See EH 2.5.2. and "Quiver representations" by Ralf Schiffler, Sections 1.1 & 5.2)

(3) Straightforward once we follow the definitions.

(2): Since we have proved that $V_i = e_i V$ is a subspace of V and φ_α takes V_i to V_j for every $\alpha: i \to j$, it remains to show that φ_α is linear $\forall \alpha \in Q_1$.

This is routine.

$$x \cdot v_i = \begin{cases} e_i \cdot v_i = v_i \\ e_j \cdot v_i = 0 \\ \varphi_\alpha(v_i) \in V_{i'} \end{cases}$$

(1): <u>Method 1.</u> check that the assignments specified satisfies the necessary relations.

<u>Method 2.</u> Define the kQ action on $V = \oplus V_i$ more generally for every path in kQ: for a path $p = \underbrace{\alpha_k \cdots \alpha_2 \alpha_1}_{\text{arrows}}$ from i to j, define

$$p \cdot (v_1, v_2, \cdots, \underset{i}{v_i}, \cdots, v_n) = (0, 0, \cdots, \underset{j\text{th spot}}{\varphi_{\alpha_k} \circ \cdots \circ \varphi_{\alpha_1}(v_i)}, 0, \cdots, 0),$$

then check this definition satisfies all the module axioms.

See EH. 2.5.2. □