Math 440. Lecture 19.

Midterm covers: Ch1 – Ch3 except 2.5.2, 3.4.1, 3.4.3, and 3.5.

03.03.2021.

Last time: · Properties of module lengths.

· Simple modules of quiver path algebras kQ. Q acyclic.

Thm 1. $\exists$ bij $S: Q_0 \longrightarrow \{\text{simple modules of } kQ\}/\text{iso}$

$$i \longmapsto S_i = Ae_i / Ae_i^{\geq 1} = \text{Span}\{e_i + Ae_i^{\geq 1}\}$$

Proved: (1) S_i is simple $\forall i \in Q_0$ (2) S is inj

Need: (3) S is surj.

$\swarrow$ will use

Lemma: $\forall i \in Q_0$, $J_i = Ae_i^{\geq 1}$ is unique maximal submodule of Ae_i.

Today: Proof of Thm 1. Quiver representations.

1. Pf of Thm 1 : Let S be a simple A-module. We need to show that S is isomorphic to $S_i = Ae_i / J_i$ for some $i \in Q_0$. Pick $0 \neq s \in S$. Then

($A = kQ$)

$$0 \neq s = 1 \cdot s = \left(\sum_{i \in Q_0} e_i\right) \cdot s = \sum_{i \in Q_0} (e_i \cdot s). \quad \text{so } \underset{S}{e_i s} \neq 0 \text{ for some}$$

$i \in Q_0$. It then follows from Lemma 3.3. (Feb. 22) that $e_i s$ generates S, ie, $S = Ae_i s$. Now consider the map $\psi : Ae_i \to S = Ae_i s$ given by $\psi(x) = xs$ $\forall x \in Ae_i$. (so ψ is right mult. by s: $x = ae_i \mapsto ae_i s$). ψ is clearly linear and a (left) module hom : $\psi(a \cdot x) = \psi(ax) = axs = a(xs) = a\psi(x)$.

("Right mult give left module homs for submodules of regular modules.") Note that ψ is clearly surj. So $S = \operatorname{Im}\psi \cong Ae_i / \ker\psi$. that is,

S is iso to a simple quotient of Ae_i. But J_i is the unique maximal submodule of Ae_i, so S_i is the unique simple quotient of Ae_i, therefore $S \cong S_i$. $\square$

E.g.: For the quiver $1 \leftarrow 2 \leftarrow \cdots \leftarrow n$ you showed $A = kQ \cong T_n(k)$ in Ex. 1.18.

You also showed that $T_n(k)$ has n simple modules up to iso. This is compatible with thm 1's prediction that kQ has n simple modules.

(**Fact:** If two algebras A_1, A_2 are isomorphic, there is a bijection between their simple modules.)

2. Quiver representations

We'll define representations of quivers.

The upshot will be: "Representations of a quiver Q are the same as repn/modules of the quiver's path algebra kQ".

We don't need to assume that Q is acyclic.

Some observations (on modules of quiver path algebras $kQ \curvearrowright V$)

$$\circ \xleftarrow{\beta} \circ \xleftarrow{\alpha} \circ$$

$$(\beta\alpha)\cdot v = \beta\cdot(\alpha\cdot v)$$

- For any kQ module V, since the set

$$Q^{\leq 1} := \{ e_i : i \in Q_0 \} \cup Q_1$$

generates kQ, to specify the kQ action it suffices to specify the action of the paths in $Q^{\leq 1}$.

- Given any kQ module V, for every vertex $i \in Q_0$ the set $e_i V$ is a subspace and in fact a submodule of V on which

$$(*) \quad p\cdot(e_i\cdot v) = \begin{cases} e_i\cdot e_i v = e_i^2\cdot v = e_i v & \text{if } p = e_i \\ e_j\cdot e_i v = e_j e_i\cdot v = 0\cdot v = 0 & \text{if } p = e_j,\ j \neq i \\ p e_i v = p v = e_j p\cdot v \in e_j V & \text{if } p \text{ is a path from } i \text{ to } j \end{cases} \quad \forall v \in V$$

Note: (HW) $V = \bigoplus_{i \in Q_0} e_i V$.

That is, V admits a collection of submodules $\{V_i : i \in Q_0\}$ related by $(*)$.

Def. A representation (over a ground field k) of a quiver $Q = (Q_0, Q_1)$ is the data $(V_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ consisting of a vector space V_i for each $i \in Q_0$ and a linear map $\varphi_\alpha : V_i \longrightarrow V_j$ for every arrow of the form $\alpha : i \longrightarrow j$ in Q_1.

E.g. (1). One loop quiver. A rep of Q is just a vector space $V = V_1$ plus an endomorphism $\varphi_\alpha : V \to V$.

Q : 

(2) $Q : 1 \xrightarrow{\alpha} 2$ A rep of Q consists of vector spaces V_1, V_2 and a linear map $\varphi_\alpha : V_1 \longrightarrow V_2$.

e.g. $\underset{V_1}{\underset{\shortparallel}{K^2}} \xrightarrow{[1 \ 1]} \underset{V_2}{\underset{\shortparallel}{K}} \rightsquigarrow \varphi_\alpha : K^2 \to K$, $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto [1 \ 1]\begin{bmatrix} x \\ y \end{bmatrix} = x+y$.

Thm. (Reps of Q $\equiv$ Reps of kQ .)

(1). (Q-reps $\to$ kQ-modules) Given a rep $(V_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ of Q, we can construct a unique kQ module V s.t. $V = \bigoplus_{i \in Q_0} V_i$ and the linear action is such that $\forall x \in Q^{\geq 1}$, $i \in Q_0$, $v_i \in V_i$,

$$x \cdot v_i = \begin{cases} e_i \cdot v_i = v_i & \text{if } x = e_i \\ e_j \cdot v_i = 0 & \text{if } x = e_j,\ j \neq i \\ \varphi_\alpha(v_i) \in V_{i'} & \text{if } x = \alpha : i \to i'. \end{cases}$$

(2). (kQ-modules $\to$ Q-reps) Given a module V of kQ, we can define a rep $(V_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ of Q by setting $V_i = e_i V$ $\forall i \in Q_0$ and $\varphi_\alpha : V_i \to V_j$ to be the map with $\varphi_\alpha(e_i v_i) = \alpha \cdot e_i \cdot v = \alpha \cdot v$ $\forall v_i \in V_i$ for each arrow $\alpha : i \to j$.

$\overset{\parallel}{(e_j \cdot \alpha) \cdot V} \in e_j V = V_j$. $\to$ ends up in the right space.

(3) The constructions in (1) and (2) are inverse to each other.

In particular, reps of Q are in bijection with modules of kQ.

Pf: next time!