# Math 4140. Lecture 18.

03.01.2021.

**Last time:** (I) Existence & Uniqueness of comp. series for finite-dim modules.
JH thm.

(II) "Length behaves well": Let $A$ be a k-alg. $V$ an $A$-module of finite length, $U \le V$ a submodule. Then

1). $V/U$ has a comp series.

2). $V$ has a comp series with $U$ appearing, and $\ell(V) = \ell(U) + \ell(V/U)$.

3). $\ell(U) \le \ell(V)$, with $\ell(U) = \ell(V)$ iff $U = V$.

**Today.** · Pf of (II)

· Simple modules of path algebras of acyclic quivers.

**1. Pf of (II).** Maintain the same notation. Recall that (2) $\Rightarrow$ (3).

We sketch the proofs of (1) and (2).

(1). Take a comp series $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ of $V$. Consider the chain

$$0 = V_0 + U/U \subset V_1 + U/U \subset \cdots \subset V_n + U/U = V + U/U = V/U \qquad (*)$$

**Claim:** $\forall\, 1 \le i \le n$, $\dfrac{(V_i + U/U)}{(V_{i-1} + U/U)} \cong V_i/V_{i-1}$ after removal of duplicate terms in $(*)$.

**Pf:** EX.

(2). By the inheritance lemma, we have a comp series $0 = U_0 \subset U_1 \subset \cdots \subset U_k = U$ of $U$.

By (1) and the correspondence thm (for any module $W$ w/ $U \le W \le V$, the (maximal) submodules of $W/U$ correspond bijectively to the (maximal) submodules of $W$ which contain $U$), we have a comp series $0 = V_0/U \subset \cdots \subset V_{\ell-1}/U \subset V_\ell/U = V/U$.

$$\overset{\displaystyle 0 = V_0/U}{\underset{\displaystyle U/U}{\shortparallel}}$$

where $U = V_0 \subset V_1 \subset \cdots \subset V_\ell = V$ are all submodules of $V$ containing $U$

and $U_i$ is a maximal submodule of $V_{i+1}$ for all $0 \le i \le n-1$.

It follows that $\underbrace{0 = U_0 \subset \cdots \subset U_k}_{\ell(U)} = U = \underbrace{V_0 \subset V_1 \subset \cdots \subset V_\ell}_{\ell(U/V)} = V$

is a comp series of $V$ with $U$ in it and $\ell(V) = \ell(U) + \ell(U/V)$.

$\square$

## 2. Classifying simple modules for path algebras of acyclic quivers.

Let $Q = (Q_0, Q_1)$ be an acyclic (no oriented cycles) quiver and let $A = kQ$ be its path algebra.

Recall that $kQ$ is f.d. and that $A$ has a basis $\mathcal{P} = \{$all paths on $Q\}$ and that we "multiply paths in the same way we multiply/compose functions".

$$\circ \xrightarrow{\alpha} \circ \xrightarrow{\beta} \qquad \beta\alpha \ ✓ \qquad \alpha\beta \ ✗$$

Note that for each $i \in Q_0$.

- The set $Ae_i$ is a submodule of $A$.
  It equals the span of all path starting at $i$ in $kQ$.
- The set $J_i = Ae_i^{\geq 1} \overset{\text{def}}{:=} \mathrm{Span}\{$path starting at $i$ of length at least $1\} \leq Ae_i$.
  It is a submodule of $Ae_i$.
- It follows that the quotient module $S_i := Ae_i / J_i$ makes sense.
  Note that $S_i = \mathrm{Span}\{e_i + J_i\}$, so $\dim S_i = 1$ and hence $S_i$ is simple.

We've now obtained a simple module $S_i$ for every vertex $i \in Q_0$.

Equivalently, we've obtained $S: Q_0 \longrightarrow \{\text{simple } A\text{-modules}\}/\text{iso}$, $i \longmapsto$ the iso class of $S_i$.

How does $A = kQ$ act on $S_i$ ?

Need $p \cdot (\underbrace{e_i + J_i}_{\text{only element in the basis of } S_i})$ for every path $p$ on $Q$.

$$p \cdot (e_i + J_i) = (pe_i) + J_i = \begin{cases} 0 & \text{if } \operatorname{source}(p) \neq i. \\ p + J_i & \text{if } \operatorname{source}(p) = i \end{cases} = \begin{cases} 0 & \text{if } \operatorname{source}(p) \neq i \\ 0 & \text{if } \operatorname{source}(p) = i \text{ and } \operatorname{len}(p) \geq 1 \\ e_i + J_i & \text{if } p = e_i. \end{cases}$$

In particular, $e_j \cdot (e_i + J_i) = \delta_{ij} e_i + J_i \quad \forall j \in Q_0$.

**Prop.** Let $i, j \in Q_0$ be distinct vertices of $Q$. Then $S_i \not\cong S_j$ as $A$-modules.

(In other words, the map $S$ is injective. )

**Pf (sketch):**

Version 1. The elt $e_j \in A$ annihilates $S_i$ but $e_i$ doesn't.

(see "Preservation of scalar actions" from Lecture 11).

Version 2. See P76 of [EH].

**Thm 1.** Every simple $A$-module is isomorphic to $S_i$ for some $i \in Q_0$.

Equivalently, the map $S : Q_0 \to \underbrace{\{\text{Simple } A\text{-modules}\}}_{\text{iso}}$ is a bijection.

$$i \longmapsto S_i$$

**Lemma:** (1) $\forall i \in Q_0$, the v.s. $e_i A e_i$ is of dim. 1 and spanned by $e_i$.

(2) $\forall i \in Q_0$, the module $J_i$ is the unique maximal submodule of $Ae_i$, and $e_i J_i = 0$.

**pf:** (1). Since $Q$ is acyclic, the only path on $Q$ that both starts and ends at $i$ is $e_i$. The statement follows since $e_i A e_i$ is the span of such paths.

(2) Note that $J_i = Ae_i^{\geq 1}$ is the span of paths starting at $i$ of length at least 1. Such path can't end at $i$ since $Q$ is acyclic, so $e_i J_i = 0$. Also note that $J_i$ is a maximal submodule of $Ae_i$ since $\dim J_i = \dim Ae_i - 1$. So it remains to show that $J_i$ is the only maximal submodule of $Ae_i$. Let $U$ be a submodule that strictly contains $U$. Then we can pick an elt $u \in U$ of the form $u = ce_i + u'$ for some $0 \neq c \in k$ and $u' \in J_i$.

But then $\underset{\in A}{e_i}\,\underset{\in U}{u} = e_i(ce_i + u') = ce_i^2 + \underbrace{e_i u'}_{\in e_i J_i} = ce_i + 0 \in U$,

hence $e_i = \left(\frac{1}{c}\right)(ce_i) \in U$. But then $Ae_i \subseteq U$, so $U = Ae_i$.

It follows that $J_i$ is a maximal submodule of $Ae_i$. $\square$

We'll see how the lemma helps the proof of Thm 1 next time.