Math 4140. Lecture 17. 02. 26. 2021.

Last time:

- examples illustrating Jordan-Hölder Thm.
- pf of existence of comp. series for f.d. modules.
- preparation for proving the Jordan-Hölder Thm.

Today.

- pf of the Jordan-Hölder Thm.
- properties of module lengths.

useful: 2nd Iso Thm: $U, W \le \overset{A \circlearrowright}{V}$

$$\begin{tikzcd}[column sep=small] & U+W \arrow[dl, dash] \arrow[dr, dash] & \\ U \arrow[dr, dash] & & W \arrow[dl, dash] \\ & U\cap W & \end{tikzcd}$$

$$\frac{U+W}{U} \cong \frac{W}{U\cap W}.$$

also. 3rd Iso Thm: $U \le W \le V$

$$\frac{V/U}{W/U} \cong V/W.$$

Pf of the JH thm: Suppose that V has comp. series

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V_n = V \qquad (\mathrm{I}).$$

and

$$0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_{m-1} \subset W_m = V \qquad (\mathrm{II}).$$

We need to show that (I) and (II) are equivalent. We do so by induction on n.

Base cases: $n=0 \Rightarrow V=0 \Rightarrow m=0$ DONE.

or $n=1$. $\Rightarrow V$ is simple $\Rightarrow m=1$. DONE.

Inductive step: Now assume $n>1$. By the inductive hypothesis, given any two two comp series $(\mathrm{I}')\ 0 = V_0' \subset V_1' \subset \cdots \subset V_{n'}' = V'$ $(\mathrm{II}')\ 0 = W_0' \subset \cdots \subset W_{m'}' = V'$ for some module V' of A and $n'<n$, then (I') and (II') are equivalent.

We prove the equivalence of (I) and (II) in two cases:

Case 1. $V_{n-1} = W_{m-1}$, say they both equal U.

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset \overset{U}{V_{n-1}} \subset V_n = V \qquad (\mathrm{I}).$$

$$0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset \underset{\underset{U}{\parallel}}{\overset{\parallel}{W_{m-1}}} \subset W_m = V \qquad (\mathrm{II}).$$

$\left(\text{Certainly } V_n/V_{n-1} = V/U = W_m/W_{m-1} \right)$

Consider the truncations $(\mathrm{I}')\ 0 = V_0 \subset \cdots \subset V_{n-1} = U$ and $(\mathrm{II}')\ 0 = W_0 \subset \cdots \subset W_{m-1} = U$.

By the inductive hypo. (since $n-1 < n$) (I') and (II') are equivalent.

It follows that (I) and (II) are equivalent.

Case 2. $V_{n-1} \neq W_{m-1}$. In this case, let $D = V_{n-1} \cap W_{m-1}$. Note that:

$$\left(\begin{array}{ll} 0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V_n = V & (\mathrm{I}). \\ 0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_{m-1} \subset W_m = V & (\mathrm{II}). \end{array} \right)$$

(i). $V_{n-1} + W_{m-1} = V$: this holds since V_{n-1} and W_{m-1} are distinct maximal submodules of V.

(ii). $V_{n-1}/D \cong \overset{V_{n-1}+W_{m-1}}{\overset{\parallel}{V}}/W_{m-1}$, $W_{m-1}/D \cong \overset{V_{n-1}+W_{m-1}}{\overset{\parallel}{V}}/V_{n-1}$: this follows from (i) and the 2nd Iso Thm. Moreover, these four quotients are simple since V/W_{m-1} and V/V_{n-1} are.

Now take a comp. series $0 = D_0 \subset D_1 \subset \cdots \subset D_t = D$ of D. (Such a series exists by the inheritance lemma)

By (i) and (ii), it induces comp series

$$0 = D_0 \subset D_1 \subset \cdots \subset D_t = D \,\dot{\subset}\, V_{n-1} \,\dot{\subset}\, V \qquad (\mathrm{III})$$

$$0 = D_0 \subset D_1 \subset \cdots \subset D_t = D \,\dot{\subset}\, \underbrace{W_{m-1} \,\dot{\subset}}\, V \qquad (\mathrm{IV})$$

of V.

$\because$ four simple quotients

Truncating (I)–(IV), we get the following four comp series, two for V_{n-1} and two for W_{m-1}:

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V' \subset V \qquad (1)$$

$$0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_{m-1} \subset V'' \subset V \qquad (2)$$

V_{n-1}

$$0 = D_0 \subset D_1 \subset \cdots \subset D_t = D \subset V_{n-1} \subset V' \qquad (3)$$

W_{m-1}

$$0 = D_0 \subset D_1 \subset \cdots \subset D_{\underset{\substack{\| \\ n-2}}{t}} = D \subset W_{\underset{\substack{n-1}}{m-1}} \subset V'' \qquad (4)$$

Applying the ind. hypo. on V_{n-1}, we get $n-1 = t+1$ (so $t = n-2$) and (1) and (3) are equivalent. Since $t = n-2$ in (4), we have $m-1 = (n-2)+1 = n-1$, so $m = n$. Now, applying the inductive hypo to (4) and (2), we see that (4) and (2) are equivalent.

Now.

1) ~ 3) $\Rightarrow$ in (I). $0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V$, the comp factors are a permutation of V/V_{n-1}, V_{n-1}/D, D_t/D_{t-1}, $\cdots$, D_2/D_1, D_1/D_0.

2) ~ 4) $\Rightarrow$ in (II): $0 = W_0 \subset W_1 \cdots \subset W_{m-1} \subset W_m = V$, the comp factors are a permutation of V/W_{m-1}, W_{m-1}/D, D_t/D_{t-1}, $\cdots$, D_2/D_1, D_1/D_0.

Since $V/V_{n-1} \cong W_{m-1}/D$ and $V_{n-1}/D \cong V/W_{m-1}$ by (ii), it follows that (I) and (II) are equivalent. $\square$

Properties of module lengths.

A: k-alg.

Recall: (1) A module V of A is said to have finite length if it has a comp series.

(2) By the JH thm, if V has finite length then we may define the length of V, written $\ell(V)$, to be the common length of all its comp. series.

E.g. We already saw that

$$\ell(V) = 0 \iff V = 0 \quad ; \quad \ell(V) = 1 \iff V \text{ is simple}.$$

More generally, $\ell(V)$ may be viewed as a measure of how far away V is from being simple.

Prop (Prop. 3.17. "Length behaves well.") Let A be a k-alg and V an A-module of finite length. Then for every submodule $U \le V$ we have:

(1). V/U has a comp series. (So quotients of finite-length modules are finite-length.)

(2). There exists a comp. seri of V with U as one of its terms. Moreover, we have $\ell(V) = \ell(U) + \ell(V/U)$.

(3). We have $\ell(U) \le \ell(V)$, and equality holds iff $U = V$.

} generalize familiar lin. alg. facts.

Pf: (sketch). (3) follows immediately from (2).

(1) & (2). $\rightarrow$ next time. (try it yourself first!)