Math 4140. Lecture 16.

Midterm I: take-home, available 5pm, March 5 – 5pm, March 6. closed book. 02. 24. 2021.

Last time: · more on simple modules

· comp. series: defs. examples, statement of main results. (A: k-alg.)

a comp. series exists and is unique for any f.d. A-module

up to permutation of comp. factors

Today · More examples of comp series.

· Proofs of the existance result. Preparation for the pf of the uniqueness result.

1. Examples.

(a). Assuming the Jordan-Hölder Thm, we have that the length of a k-module ($\equiv k$-vec. space) is just $\dim_k V$.

'A

(b). $\underset{A}{\underline{M_n(k)}} \curvearrowright \underset{A=V}{\underline{M_n(k)}} = C_1 \oplus C_2 \oplus \cdots \oplus C_n$. (eg. 3.7.(4) / 3.14. 1))

Two distinct composition series: $\quad C_i = \left\{ \begin{bmatrix} 0 & \underset{i}{\left|\begin{array}{c} * \\ * \\ \vdots \\ * \\ \vdots \end{array}\right|} & 0 \end{bmatrix} \right\} \underset{HW.}{\overset{\sim}{=}} k^n = \left\{ \begin{bmatrix} * \\ \vdots \\ * \end{bmatrix} \right\}$.

① Let $V_i = C_1 \oplus C_2 \oplus \cdots \oplus C_i \quad \forall 1 \leq i \leq n$.

Then the chain $0 =: V_0 \subset V_1 \subset V_2 \cdots \subset V_n = V$ is a comp series of V, the quotients are $V_i / V_{i-1} \cong C_i \cong k^n$, which is simple as an A-mod.

② Let $W_i = C_n \oplus C_{n-1} \oplus \cdots \oplus C_{n-i+1} \quad \forall 1 \leq i \leq n$.

Then the chain $0 =: W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_n = V$ is a comp series of V for the same reason, this time with $W_i / W_{i-1} \cong C_{n-i+1} \cong k^n$.

① and ② give us two distinct but equivalent comp series of V, each with n comp. factors that are all iso to k^n.

(c). $A = k \times k$ @ $V = A$. $S_1 = \{ (x, 0) : x \in K \} \subseteq A$, $S_2 = \{ (0, y) : y \in k \} \subseteq A$.

(eg. 3.14. (3))

Ex: (1). S_1 and S_2 are submodules of A. Moreover, they are simple.

(2). $A/S_1 \cong S_2$, $A/S_2 \cong S_1$.

(3) $S_1 \ncong S_2$.

It follows that

$$0 =: V_0 \subset V_1 = S_1 \subset V_2 = V = A$$

$$0 =: W_0 \subset W_1 = S_2 \subset W_2 = V = A$$

are both comp. series of A, with quotients $\left(V_1/V_0 \cong S_1 , V_2/V_1 \cong S_2 \right)$ and $\left(W_1/W_0 \cong S_2 , W_2/W_1 \cong S_1 \right)$. So the seq. of comp factors are not the same up to iso, but the series are still equivalent.

(after a permutation of comp factors.)

2. Proofs

Let A be a k-algebra.

We already saw that not every A-module has a comp series $\left(A=k \circlearrowright V, \dim V = \infty\right)$.

But finite-dimensional A-modules do always have comp. series:

Prop: (Lemma 3.9) Every f.d. A-module V has a comp series.

Pf: Use induction on $d := \dim_k V$.

Base case: $d=0$ or $d=1$. $\xrightarrow{\text{last lecture}}$ V has a comp series of length 0 or 1, resp. ✓.

Inductive step: Say $d>1$. If V is simple, then again we saw that V has a comp series of length 1. So assume V is not simple, i.e., V has at least one nonzero proper submodule. Take a maximal nonzero proper submodule U of V. By induction, since $\dim U < \dim V = d$, U must have a comp series $0 =: U_0 \subset U_1 \cdots \subset U_l = U$. (*)

(*) U_i/U_{i-1} is simple $\forall\, 1 \le i \le l$

But then $0 = U_0 \subset U_1 \cdots \subset U_l = U \subset V$ is a comp series of V: $\begin{cases} V/U \text{ is simple since } U \subseteq V \text{ is max.} \\ U_i/U_{i-1} \text{ is simple by (*).} \end{cases}$ □

So the existence part is easy: uniqueness is harder:

Thm. (Jordan-Hölder, Thm 3.11) Suppose an A-module V has two comp. series

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V \qquad (I)$$

$$0 = W_0 \subset W_1 \subset \cdots \subset W_{m-1} \subset W_m = V \qquad (II).$$

Then they are equivalent, i.e., $n = m$ and there is a permutation $\sigma \in S_n = S_m$ s.t. $V_i/V_{i-1} \cong W_{\sigma(i)}/W_{\sigma(i-1)}$ $\forall\ 1 \leq i \leq n$.

Rmk: The proof will mainly be an exercise in induction and the second isomorphism theorem for modules.

$H \leq G$, $K \trianglelefteq G$

for gps:

for modules

$u, w \leq V$

$$\begin{tikzcd}
 & u+w \arrow[dl, dash] \arrow[dr, dash] & \\
u \arrow[dr, dash] & & w \arrow[dl, dash] \\
 & u\cap w &
\end{tikzcd}$$

$\rightsquigarrow$ $u+w/u \cong w/u\cap w$

$u+w/w \cong u/u\cap w$.

module isos.

Lemma (inheritance. Prop 3.10.) If V has a comp series, then every submodule $U \le V$ also has a comp series.

Pf: Take a comp series $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ of V.

Intersecting each term in the series w/ U yields a chain of subspaces

$$0 = V_0 \cap U \subset V_1 \cap U \subset \cdots \subset V_n \cap U = V \cap U = U \quad (*)$$

Removing duplicate terms if necessary, we may assume that all containment in $(*)$ are strict. We claim that $V_i \cap U / V_{i-1} \cap U \underset{\urcorner}{\cong} \underbrace{V_i / V_{i-1}}_{\text{simple.}}$, so $(*)$ is a comp series.

Pf of the claim that $V_i \cap U / V_{i-1} \cap U \cong V_i / V_{i-1}$:

(i). By the 2nd Iso Thm
$$\left(\begin{tikzcd} & W_1+W_2 \arrow[dl, dash] \arrow[dr, dash] & \\ W_1 \arrow[dr, dash] & & W_2 \arrow[dl, dash] \\ & W_1 \cap W_2 & \end{tikzcd}\right)$$
, we have

$$(V_i \cap U)/(V_{i-1} \cap U) = (V_i \cap U)/V_{i-1} \cap (V_i \cap U) \cong \frac{V_{i-1}+(V_i \cap U)}{V_{i-1}} \subseteq V_i / V_{i-1}$$

(ii). Since $V_{i-1} \cap U \neq V_i \cap U$ by assumption,

$$V_i \cap U / V_{i-1} \cap U \neq 0$$

Since V_i / V_{i-1} is simple, (i) and (ii) imply that $V_i \cap U / V_i \cap U \cong V_i / V_{i-1}$. □

Next time: pf of the Jordan-Hölder Thm.