Math 4140. Lecture 15.

Midterm? — Fri. March 5.

02.22.2021.

Last time:

- From gp actions to modules/reps of gp algebras.

$$G \curvearrowright X \longrightarrow A = kG \curvearrowright V = kX.$$

$$(g, x) \longmapsto g \cdot x \qquad\qquad g \cdot e_x = e_{g \cdot x}$$

- Simple modules : nonzero modules w/ exactly two submodules

— basic examples : modules of dim. 1 are simple

$M_n(k) \curvearrowright k^n$ is simple.

$S_n \curvearrowright [n], n > 1 \longrightarrow kS_n \curvearrowright k[n]$: not simple

Today: · more on simple modules · Composition series.

1. More on simple modules.

Warm-up examples.

(1). $A = k$. An A-module is just a k-vector space V, and V is simple iff $\dim_k V = 1$.

(2) Let D be a division algebra (so every nonzero elt is invertible).

The regular module $D \circlearrowright D$ is simple: Let $U \subseteq D$ be a submodule that's not zero. Need to show that $U = D$. Since $U \neq 0$, there is a nonzero elt $u \neq 0$ in U. Now, u^{-1} exists in D since D is a division algebra, so $u^{-1} \cdot u = 1 \in U$. But then $\forall d \in D$, $d = d \cdot 1 \in U$, so $U = D$. $\square$

Lemma 3.3. ("Cyclicity test for simpleness") Let A be a k-alg and V a nonzero A-module. Then V is simple if and only if $\forall v \in V \setminus \{0\}$ we have $Av = V$.

Pf: ($\Rightarrow$). Let $v \in V \setminus \{0\}$. Recall that Av is a submodule of V. Thus, if V is simple, then $Av = 0$ or $Av = V$. But $v = 1 \cdot v \in Av$, so $Av = 0$, therefore $Av = V$.

($\Leftarrow$). Suppose $Av = V$ $\forall v \in V \setminus \{0\}$. Let $U \subseteq V$ be a submodule that is not zero. Take $u \in U$ nonzero. Then $Au = V$ by supposition. But $Au \subseteq U$, so $V \subseteq U$, so $U = V$. It follows that V is simple. $\square$

Lemma 3.4. (simplicity of quotient modules) Let A be a K-alg, V an A-module and $U \subseteq V$ a proper submodule. Then V/U is simple iff U is a maximal submodule of V (i.e., if W is a submodule of V with $U \subseteq W \subseteq V$ then $W = U$ or $W = V$)

Pf: This follows from the correspondence thm for modules.

$$\left\{ \begin{array}{c} \text{submodules of} \\ V/U \end{array} \right\} \xleftrightarrow{\text{bij}} \left\{ \begin{array}{c} \text{submodules of } V \\ \text{containing } U \end{array} \right\}$$

$$W/U \longleftarrow\!\!\shortmid \; W$$

$\square$

2. Composition Series.

Definitions. Let A be a k-alg and let V be a A-module.

- (composition series. length) A composition series of V is a finite chain of A-modules (*) $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V$ s.t. V_i/V_{i-1} is simple $\forall\, 1 \leq i \leq n$.

  $\Updownarrow$ V_{i-1} is a max. submodule in V_i

  The length of such a series is n, the number of quotients.
- (finite length) We say V has finite length if it admits a comp. series.
- (terms, composition factors) In a comp. series of the form (*), we call the modules $V_0, V_1, \dots, V_n$ the terms of the series, and we call the simple quotients $V_1/V_0, V_2/V_1, \dots, V_n/V_{n-1}$ the composition factors.
- (equivalence) Two composition series of V are called equivalent if they have the same length and their composition factors are the same / iso and reordering.

The main results on composition series are: Let A be a k-alg.

(1). Existence. (Lemma 3.9.) Every f.d. A-module has a composition series (and therefore has finite length).

(2) Uniqueness (Thm 3.11. Jordan-Hölder Thm.) Suppose an A mod V has two composition series. Then they are equivalent.

Def: If V has finite length, then we define the <u>length of V</u> length(V) to be the length of any comp series of V. If V is not of finite length, we say the length of V is infinite.

more examples and pfs next time.

Before we prove the result, some examples:

(1) the zero module $V=0$. $0=V_0=V$ is the unique comp series of V, so $\text{length}(0)=0$.

(2) simple modules V. $0=V_0\subset V_1=V$ is a comp series since $V_1/V_0\cong V_1$ is simple. So $\text{length}(V)=1$.

e.g. $M_n(k) \curvearrowright k^n$ simple.
$\text{length}(k^n)=1$, $\dim_k(k^n)=n$.

(3) $A=k$, V = an inf. dim. k-v.s. $\rightarrow$ V has no comp. series.

Note that V has no comp. series: if it did, we'd have a finite chain $0=V_0\subset V_1 \cdots \subset V_n=V$ with $\dim(V_i/V_{i-1})=1$ $\forall i$.

This would imply $\dim V=n<\infty$ (Fact: $\dim V/U=\dim V-\dim U$.)