Math 4140. Lecture 14.

02.19.2021

Last time: · Modules of $k[x]$ and $k[x]/\langle f\rangle$

$\forall$ v.s. V, $\quad \mathrm{End}_k(V) \xrightarrow{\text{bij}} k[x]\text{-modules}$

$\alpha \longmapsto V_\alpha,\ x\cdot = \alpha \rightsquigarrow$ naturally a $k[x]/\langle f\rangle$ module with

$\bar{x} := x + \langle f\rangle$ acting as α

iff $f(\alpha) = 0$

Today. · Group actions vs. modules of gp algebras.

· Simple modules

2. Gp actions vs. modules/reps of gp algebras

$$G \times X \longrightarrow X$$

Main goal: Show that any gp action $G \circlearrowright X$, $(g,x) \mapsto g.x$

naturally induces a kG-module $^{kG}\circlearrowright kX$ with $\underline{g\,x} = g.x$.

as usual, it suffices to specify how a basis elt of the alg acts on a basis elt of the module.

E.g. $G = S_3 \circlearrowright X = \{1, 2, 3\}$

$$\rightsquigarrow$$

$$A = kG = kS_3 \circlearrowright V = kX = k\langle e_1, e_2, e_3 \rangle.$$

$$\boxed{\mathrm{End}_k(V) \overset{\text{fix basis } \{e_1,e_2,e_3\}}{=\!=} M_3(k)}$$

Take $\underline{a = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}} \in G$ $\xrightarrow{\text{"upgrade"}}$ the linear map f_a: $e_1 \mapsto e_1$, $e_2 \mapsto e_3$, $e_3 \mapsto e_2$

a function

$$a: \begin{array}{l} 1 \to 1 \\ 2 \to 3 \\ 3 \to 2 \end{array}$$

$$\Bigg/\!\!\Bigg/$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Details: Let k be a field. G a gp. Recall that:

- A (left) <u>group action</u> of G on a set X is a map $G \times X \to X$, $(g,x) \mapsto g.x$

s.t. (a) $g.(h.x) = (gh).x$ $\forall g,h \in G, x \in X$ (b) $1_G \cdot x = x$ $\forall x \in X$.

E.g. $S_n \circlearrowright [n]$ (a). $g.(h.x) = g(h(x)) = (g \circ h)(x) = (gh).x$; (b) $1_G . x = id_X(x) = x$.
$\overset{\shortparallel}{\{1, 2, \ldots, n\}}$

- Given any set X, we may form it free vec. space $V = kX$.

From the gp G, we may also form its gp algebra $A = kG$.

Hope: Upgrade the gp action $G \circlearrowright X$ to a module action $A = kG \circlearrowright kX = V$,

Will: construct a rep (i.e., an alg hom) $\theta : A = kG \longrightarrow \mathrm{End}_k(V) = \mathrm{End}_k(kX)$.

$a \in g \longmapsto \theta(a) : kX \to kX$.

Notation: We write $kX = \mathrm{Span}_k\{e_x : x \in X\}$.

How? (1) specify a linear map $f_a \in \operatorname{End}_k(kX)$ $\forall a \in G$:

let $f_a(e_x) = e_{a \cdot x}$ $\forall x \in X$ and extend linearly

(2) extend the assignment $a \mapsto f_a$ from (1) linearly to a linear map

$\theta : kG \longrightarrow \operatorname{End}_k(kX)$ $\quad a \longmapsto \theta(a) = f_a$

(3) show that θ is multiplicative, hence an alg hom, hence a rep of kG.

Need: $\theta(gh) = \theta(g)\,\theta(h)$ $\forall g,h \in G$, $\theta(1_G) = id_{kX}$

i.e. $f_{gh} = f_g f_h$ $\forall g,h \in G$. It suffices to show

(a) $f_{gh}(e_x) = f_g f_h(e_x)$ $\forall x \in X$, (b) $f_{1_G}(e_x) = e_x$ $\forall x \in X$.

$(*)\left\{\begin{array}{l} f_{gh}(e_x) = e_{gh \cdot x} = e_{g \cdot h \cdot x} = f_g \cdot f_h(e_x) \ \forall x \in X. \\ \\ f_{1_G}(e_x) = e_{1_G \cdot x} = e_x . \end{array}\right.$

gp action axioms!

Alternative way to get the rep $\theta: kG \to \mathrm{End}(kX)$.

(1) (same as before) specify a linear map $f_a \in \mathrm{End}_k(kX)$ $\forall a \in G$:

let $f_a(e_x) = e_{a\cdot x}$ $\forall x \in X$ and extend linearly.

Doing so defines a map $f: G \to \mathrm{End}_k(kX)$, $a \mapsto f_a$.

(2) Use univ. prop. of gp algebras (see P8 of 01.27.pdf) to upgrade

$f: G \to \mathrm{End}_k(kX)$ to an alg. hom $\theta: kG \to \mathrm{End}_k(kX)$:

$$\begin{tikzcd} & a \arrow[r, mapsto] & f_a \\ \underline{a} \arrow[dd, mapsto] & f: G \arrow[r] \arrow[d, hook, "i"'] & \mathrm{End}_k(kX) \\ & \theta': kG \arrow[ur, dashed, "\theta"'] & \\ a & & \end{tikzcd}$$

Since f is a monoid hom by (a), (b) on the last page (by the same proof), it induces an alg hom $\theta: kG \to \mathrm{End}_k(kX)$ with

$$\theta(a) = \theta(i(a)) = f(a) = f_a \quad \forall a \in G.$$

yielding the same map θ as on the last page. □

2. Simple modules

$\xrightarrow{\text{will be}}$ "building blocks" of modules. Let A be a k-alg.

Def. An A-module V is called <u>simple</u> if $V \neq 0$ and V has no proper, nontrivial submodule.

E.g.

- Modules of dim 1. Let V be an A-mod with $\dim_k(V) = 1$. A submodule W of V is first of all a subspace, hence W satisfies $\dim(W) = 0$ or $\dim(W) = 1$, in which cases $W = 0$ or $W = V$, respectively. This implies that V is simple.

- $\underset{A}{\underset{\shortparallel}{M_n(k)}} \circlearrowright \underset{V}{\underset{\shortparallel}{k^n}}$ (02, 05) We saw that k^n is a simple $M_n(k)$ module.

Non-e.g. $G = S_n \curvearrowright X = \{1, 2, \cdots, n\} \longrightarrow kS_n \curvearrowright kX = k^n$. $n > 1$

We claim that kX is not a simple kS_n module. For a nontrivial, proper submodule of $V = kX$, consider $W := \mathrm{Span}\{\sum_{i=1}^n e_i\}$. Note that

$$a \cdot \left(c \cdot \sum_{i=1}^n e_i\right) = c\left(a \cdot \sum_{i=1}^n e_i\right) = c \sum_{i=1}^n a \cdot e_i = c \sum_{i=1}^n e_{a(i)} = c \sum_{j=1}^n e_j \quad \forall a \in G,\ c \in k.$$

e.g. $n = 3$. $a = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}$.

$$a \cdot (e_1 + e_2 + e_3) = a \cdot e_1 + a \cdot e_2 + a \cdot e_3 = e_1 + e_3 + e_2 = e_1 + e_2 + e_3.$$

We've show that $a \cdot W \subseteq W$ $\forall a \in G$, so W is a submodule of V. $\square$

Lemma 3.3. next time.