Math 4140. Lecture 13

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CU REU: math.colorado.edu/~thienn/InternalMREU/MathREU.html

Last time:

- Def. of algebra representations $\theta: A \to \mathrm{End}_k(V)$, $a \mapsto \theta(a)$

  $\theta(a)$: the action of a. V: the module.

- "representations $\equiv$ modules": both are about a v.s. and actions of the algebra elts.

- modules / reps of $\mathcal{P} = k[x]$.
  - any v.s. V can be made a $\mathcal{P}$-module
  - each module structure on V is determined uniquely by the action of x, because x generates $\mathcal{P}$ as an algebra.
  - in fact, for any $\alpha \in \mathrm{End}_k(V)$, we can make V a $\mathcal{P}$-module with $x \cdot v = \alpha(v)$ $\forall v \in V$. We'll denote this module by V_α.

Today. 1. modules of $k[x]/(f)$.

Take an $k[x]$-module V_α for some $\alpha \in \mathrm{End}_k(V)$. $\big(x \cdot - = \alpha(-) \big)$

Take an ideal $I \subseteq k[x]$, necessarily of the form $I = f$ for some $f \in k[x]$.

Q: Under what condition is V_α naturally a $k[x]/(f)$-module with the action $\big(a + \langle f \rangle\big) \cdot v = a \cdot v \quad \forall a \in k[x], v \in V$.

Prop. The above proposed action makes V_α a $k[x]/\langle f \rangle$-module

iff $f(\alpha) = 0$ in $\mathrm{End}_k(V)$.

Eg. $f = x^n - 1$. $\quad I = \langle x^n - 1 \rangle$.

The prop says that for $\alpha \in \mathrm{End}_k(V)$, we can make V_α an $k[x]/\langle x^n-1\rangle$ module with $(\underbrace{x + \langle x^n - 1\rangle}_{\overline{x}}) \cdot v = \alpha(v)$ iff $f(\alpha) = \alpha^n - 1 = 0_V$.

Why? "only if": Note that $(\overline{x}^n - 1) = (x^n - 1) + \langle x^n - 1 \rangle = 0$.

$1 + \langle x^n - 1 \rangle = 1_{k[x]/\langle x^n-1\rangle}$.

Therefore

$\forall v$, $\quad 0 = (\overline{x}^n - 1) \cdot v = (\alpha^n - 1) v$ since $\overline{x} \cdot v = \alpha \cdot v$

So $\alpha^n - 1 = 0$ in $\mathrm{End}_k V$.

"if": **Claim:** If $\alpha^n - 1 = 0_V$, then $(a + \langle x^n - 1 \rangle) \cdot v = a \cdot v$ is well-defined and satisfies the necessary module axioms "by inheritance".

Pf: well-definedness: suppose $a + \langle x^n - 1 \rangle = b + \langle x^n - 1 \rangle$ (1).

need $a \cdot v = b \cdot v$ (2).

$$(1) \Rightarrow a - b \in \langle x^n - 1 \rangle \Longrightarrow a \cdot v = \Big(b + \underset{k[x]}{g}(x^n - 1)\Big) \cdot v$$

$$= b \cdot v + g(x^n - 1) \cdot v$$

$$= b \cdot v + 0 \quad \text{since } (x^n - 1) \cdot v = \underset{0}{\underbrace{(\alpha^n - 1)}}(v)$$

$$= b \cdot v$$

$\Rightarrow$ (2) holds. ✓

The axioms: HW.

Note: Everything on the last two pages generalize from $f = x^n - 1$ to arbitrary f. (EH. 2.2.)

Put another way:

Let V be a vector space and $A = k[x]$. Take $f \in k[x]$.

There's a bijection

$$\mathrm{End}_k(V) \longleftrightarrow \{A\text{-mod structures on } V\}/\text{iso}$$

$$\alpha \longmapsto V_\alpha.$$

Moreover, this bijection restricts to a bijection

$$\{\alpha \in \mathrm{End}_k(V) \mid f(\alpha) = 0\} \longleftrightarrow \left\{ \begin{array}{l} A\text{-mod structures on } V \text{ that} \\ \text{naturally induce } k[x]/\langle f \rangle\text{-modules} \end{array} \right\} \Big/ \text{iso}.$$

2. Gp actions vs. modules/reps of gp algebras

Main goal: Show that any gp action $G \circlearrowright X$, $(g,x) \mapsto g.x$

naturally induces a kG-module $kG \circlearrowright kX$ with $g\,x = g.x$.

as usual, it suffices to specify how a basis elt of the alg acts on a basis elt of the module.

E.g. $G = S_3 \circlearrowright X = \{1, 2, 3\}$

$$A = kG = kS_3 \circlearrowright V = kX = k\langle 1, 2, 3\rangle. \qquad \mathrm{End}_k(V) \underset{\{e_1,e_2,e_3\}}{\overset{\text{fix basis}}{=\!=}} M_3(k)$$

Take $\underbrace{\sigma = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}}_{\text{a function}} \in G \xrightarrow{\text{"upgrade"}}$ the linear map $\bar{\sigma}$: $e_1 \mapsto e_1$, $e_2 \mapsto e_3$, $e_3 \mapsto e_2$

$$\parallel$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The proof: next time. (Try it yourself first!)