Math 4140. Lecture 12.

02.11.2021.

Last time: · module homs and isos.

· mentioned a fact about modules of $A = \mathbb{C}[x] / \langle x^n - 1 \rangle$:

$\forall \lambda \in U := \{ \lambda \in \mathbb{C} : \lambda^n - 1 = 0 \}$, there is a 1-d module $V_\lambda \overset{=k}{}$ of A s.t. $x + \langle x^n - 1 \rangle =: \bar{x} \cdot 1 = \lambda \cdot 1$.

· In fact, there is a bijection $U \longrightarrow \{1\text{-d modules of } A\} / \text{iso}$

$\lambda \longmapsto V_\lambda$.

Today: · Working towards modules of $k[x]$ and its quotients. §2.2

· Representations of algebras. Representations vs. modules §2.4.

1. Representations.

Let A be a k-algebra.

Def. (Representations of an algebra ; Def. 2.31)

A <u>representation</u> of A over k consists of a k-v.s. V and a k-algebra homomorphism $\theta : A \to \mathrm{End}_k(V) \cong M_n(k)$

Note: Once we fix a basis of V, we can identify $\mathrm{End}_k(V)$ with $M_n(k)$ via the map $f \mapsto [f]_\beta^\beta$, so a representation of A can be identified as a pair (V, θ') where V is a k-v.s and θ' is an algebra hom $\theta' : A \to M_n(k)$.

· (actions, actions, actions) Given a rep. of A $\theta : A \to \mathrm{End}_k(V)$, for each $a \in A$ we have $\theta(a) \in \mathrm{End}_k(V)$, so $\theta(a)$ acts on V as a linear map.

we can think of this action as an "action of a", similar to how a world act on a module. Point: rep $\to$ actions on V. by elts of A.

→ the same! we'll see

An important equivalence.

Thm 2.33. Let A be a k-algebra.

(1). Suppose $(V, \theta : A \to \mathrm{End}_k(V))$ is a repn of A, then the action defined by , ie., the map $A \times V \to V$ defined by

$$a \cdot v = \underbrace{\theta(a)}_{\in\, \mathrm{End}_k(V)}(v) \in V$$

makes V an A-module.

(2). Given an A-module V with an action $A \times V \to V$, $(a,v) \mapsto a.v$,

the pair $\left(V, \theta : A \to \mathrm{End}_k(V)\right)$ where

$$\theta(a)(v) = a \cdot v$$

is a rep of A.

(3) The constructions $\{\text{repns of } A\} \underset{(2)}{\overset{(1)}{\rightleftarrows}} \{A\text{-modules}\}$ are mutual inverses.

Pf: Let's prove (1). Given: rep $A \xrightarrow{\theta} \mathrm{End}_k(V)$, Constructed: $A \times V \to V$

$$a.v = \theta(a)(v).$$

We need to check the module axioms.

(i) $a \cdot (v+w) = \theta(a)(v+w) \overset{\theta(a) \text{ is lin}}{=\!=} \theta(a)(v) + \theta(a)(w) = a.v + a.w \quad \forall v, w \in V.$

(ii). $(a+b)\cdot v = \theta(a+b)\cdot v \overset{\theta \text{ is lin}}{=\!=} [\theta(a)+\theta(b)](v) = \theta(a)(v) + \theta(b)(v) = a\cdot v + b\cdot v$

(iii). $(ab)\cdot v = \theta(ab)\cdot v \overset{\theta \text{ is mult.}}{=\!=} [\theta(a)\circ\theta(b)]v = \theta(a)\big(\theta(b)(v)\big) = a\cdot(b\cdot v)$

(iv). $1_A \cdot v = \theta(1_A)v \overset{\theta \text{ is unital}}{=\!=} 1_{\mathrm{End}_k(V)}(v) = id_V(v) = v$

$\forall\, a,b \in A,\ v \in V.$

By (i) — (iv), V is an A-module.

HW: Prove (2) and (3).

Thm: " R-modules $\equiv$ R-rep " . (Change V to M and erase k from Thm 2.33; everything still holds)

Note: We've done something differently from before: we directly treated k-algebras instead of more general rings. But the def/thm generalize to rings.

Def: A rep of a ring R is a pair (M,θ) where M is an abelian gp and θ is a ring hom $\theta: R \to \mathrm{End}(M) := \{\text{gp homs } f: M \to M\}$.

Example/Application. Induced actions, revisited.

Recall that if A, B are algebras, V a B-module, and $\varphi: A \to B$ alg hom.

$A \xrightarrow{\varphi} B \curvearrowright V$. We proved that V is also an A-module via the action $a \cdot v = \underbrace{\varphi(a)}_{\in B} \cdot (v)$ by checking the module axioms.

But given Thm 2.33, no direct verification is necessary.

V is a B-module $\Longrightarrow$ the map $\theta: B \to \mathrm{End}_k(V)$, $b \mapsto (b\cdot)$, the action of b is an algebra hom

$\Longrightarrow$ the map $\theta': A \xrightarrow{\varphi} B \xrightarrow{\theta} \mathrm{End}_k(V)$, $a \mapsto \theta(\varphi(a)) = (\varphi(a)\cdot)$, the action of $\varphi(a)$ is an alg. hom

$\Longrightarrow$ V is an A-module under $a \cdot v = \theta'(a)(v) = \varphi(a) \cdot v$!!!

2. Modules of $k[x]$ and its quotients.

General observation: The R-module axioms "$(r+s)\cdot m = r\cdot m + s\cdot m$" and "$(rs)\cdot m = r\cdot(s\cdot m)$" implies that knowing the action of a generating set of R on M is enough for knowing the action of all elts of R on M. (M an R-module).

Special case: The algebra $A = k[x]$ is generated by x (and 1, which has to act as identity on any module), to specify/know the A-action on an A-module M, it suffices to specify/know the action of x.

Conversely, given any vector space V and any linear map $\alpha : V \to V$,

we may in fact make V an $k[x]$ module s.t.

$$x \cdot v = \alpha(v).$$

Note: · (Uniqueness & formula) If such a module exists, then it's unique because the action of x determines the action of all of A.

e.g. $x \cdot v = \alpha(v) \longrightarrow (2x^2 - x + 3) \cdot v = 2x^2 \cdot v - x \cdot v + 3 \cdot v$

$$= 2\alpha^2(v) - \alpha(v) + 3v$$

· (Existence) Why does such a module always exist.

Two proofs: (1) Prove that there is a module $V_\alpha = V$ s.t. $\underset{\in k[x]}{f(x)} \cdot v \overset{*}{=} f(\alpha)(v)$; in particular, $x \cdot v = \alpha(v)$; see p. 33. (2). Universal property.

A universal property proof.

Prop: Let V be a k-vec. space. For any $\alpha \in \mathrm{End}_k(V)$ we can make V a $k[x]$-module s.t. $x \cdot v = \alpha(v)$.

Pf: (Easy fact: $k[x] \cong k\langle x \rangle$ via $\varphi : x \longmapsto x$) By Thm 2.33, it suffices to construct an alg hom $\theta : k[x] \longrightarrow \mathrm{End}_k(V)$ with $\theta(x) = \alpha$. Recall that $k\langle x \rangle$ is the free unital algebra on the set $X = \{x\}$ (HW 2.(8)). Thus, the set function $f : X \to \mathrm{End}_k(V)$ with $f(x) = \alpha$ induces a unique algebra hom $\bar{f}$ with

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "\iota"'] & \mathrm{End}_k(V) \\ k\langle x \rangle \arrow[ur, dashed, "\exists ! \bar{f}"'] & \end{tikzcd}$$

$x \longmapsto \alpha$, $x \mapsto x$

$\bar{f} \circ \iota = f$, $\Rightarrow \bar{f}(x) = \bar{f}(\iota(x)) = f(x) = \alpha$.

Composing $\bar{f}$ with φ gives the desired hom
$\theta = \bar{f} \circ \varphi : k[x] \to k\langle x \rangle \to \mathrm{End}_k(V),\ x \mapsto \alpha$. $\square$