

Change to HW3. (b) : Ex. 2.2) \rightarrow Ex 2.2). (a) - (b).

Last time : • Def of module homs • "Induced" modules, direct sums/products

Today : • More on module homs/isos. • Some modules of $k[x]$ and its quotient

1. Properties of module homs & isos. Let R be a ring.

Recall (def) : A map $\varphi: M \rightarrow N$ of R -modules is a gp hom

that respects the R -actions, i.e., a map s.t.

$$(i) \quad \varphi(m_1 + m_2) = \varphi(m_1) + \varphi(m_2) \quad \forall m_1, m_2 \in M$$

$$(ii) \quad \varphi(r \cdot m) = r \cdot \varphi(m) \quad \forall r \in R, m \in M.$$

Remarks.

- (1). "that recurring theme". If $R=A$ is a k -algebra, then an A -module hom is automatically a linear map, that is, in addition to (i) it respects scaling: $\varphi(c \cdot m) = c \cdot \varphi(m) \quad \forall c \in k, m \in M$. **Pf:** EX. Use $k \rightarrow A, c \mapsto c \cdot 1_A$.
- (2) To check a map $\varphi: M \rightarrow N$ between two R -modules is a module hom, it suffices to check that φ is a gp hom and respects the R -actions.

eg (Inclusion & projection) M an R -module. $U \subseteq M$ a submodule.
 \downarrow feel free to quote results from gp theory.

The maps $\iota: U \rightarrow M, u \mapsto u; \quad \pi: M \rightarrow M/U, m \mapsto m+U$

are both R -module homs.

$$\begin{array}{ccccc} m & \xrightarrow{r \cdot} & r \cdot m & \xrightarrow{\pi} & r \cdot m + U \\ & \searrow \pi & & \swarrow r \cdot & \\ & & m+U & \xrightarrow{r \cdot} & r \cdot (m+U) \end{array}$$

\parallel def of M/U .

Check: gp hom? \checkmark π \checkmark by gp theory; respect actions? easy to check as well.

Isomorphism Thms for modules

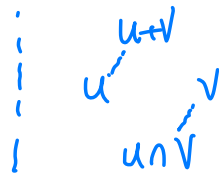
Thm. (Thm. 2.24). Let R be a ring, M, N be R -modules, and $U, V \subseteq M$ be submodules of M .

(a) If $\varphi: M \rightarrow N$ is an R -module hom. Then $\text{Im } \varphi \subseteq N$ is a submodule of N , $\ker \varphi$ is a submodule of M , and we have a well-defined module iso

$$M / \ker \varphi \longrightarrow \text{Im } \varphi, \quad m + \ker \varphi \longmapsto \varphi(m).$$

(b) The sum $U+V$ is an R -module, the intersection $U \cap V$ is an R -module,

and we have
$$U / U \cap V \cong (U+V) / V.$$



(c) Suppose $U \subseteq V$. Then V/U is an R -submodule of M/U ,

and we have
$$\frac{(M/U)}{(V/U)} \cong M/V \quad \text{as } R\text{-modules.}$$

Pf. Hw.

Internal/external direct sums.

In Hw4, you are asked to show the last claim from Lecture 10.

M : R -module. $(U_i)_{i \in I}$, each U_i a submod of M .

external
↙

↘ internal way

Can create the ext.

dir. sum $\overset{\text{E.}}{\oplus} U_i$

there's a chance that M equals the internal direct sum $\overset{\text{I.}}{\oplus} U_i$, i.e. that

$(U_i)_{i \in I}$ satisfy the conditions in

Def 2.15 (b)

(claim: If so, $\overset{\text{E.}}{\oplus} U_i \rightarrow M$, $(u_i)_{i \in I} \mapsto \sum_{i \in I} u_i$
gives an R -mod iso.

Preservation of scaling actions.

Prop. Let A be a k-alg. and $\varphi: M \rightarrow N$ an A -module hom. Suppose that some elt $r \in A$ acts as a scalar on a certain elt $m \in M$, say

$r \cdot m = cm$ for some $c \in k$. Then r must also act as the same scalar on $\varphi(m)$, i.e., $r \cdot \varphi(m) = c\varphi(m)$.

Pf: $r \cdot \varphi(m) = \varphi(r \cdot m) = \varphi(cm) = c\varphi(m)$.

The simple prop is quite useful for distinguishing non-isomorphic A -modules:

given M, N modules, if some $r \in A$ acts ^{as} a scalar c on some $m \in M$

but acts as c on no elt of N , then $M \not\cong N$.

application:

Reason for fact: next time

Fact: Let $A = \mathbb{C}[x]/(x^n - 1)$ and $U = \{x \in \mathbb{C} : x^n = 1\}$.

For all $\lambda \in U$, there is a simple 1-dim A -mod V_λ

st. $V_\lambda = \mathbb{C}$ as v.s. and $\bar{x} \cdot 1 = \lambda \cdot 1$. ($\bar{x} = x + (x^n - 1)$)

Note: $(*)$ implies that \bar{x} acts as λ on all of V_λ :

$$\forall c \in V_\lambda = \mathbb{C}, \bar{x} \cdot c = \bar{x} \cdot (c1) = c(x \cdot 1) = c\lambda = \lambda c.$$

By the prop, since \bar{x} acts on all of V_λ as λ and on all of V_μ as μ for all $\mu, \lambda \in U$, so if $\mu \neq \lambda$ then $V_\lambda \not\cong V_\mu$.

