Math 4140. Lecture 11

02.10.2021

Change to Hw3.(6) : Ex. 2.21 $\longrightarrow$ Ex 2.21. (a) – (b).

Last time: · Def of module homs · "Induced" modules, direct sums/products

Today. · More on module homs / isos. · Some modules of $k[x]$ and its quotient

1. Properties of module homs & Isos.

Let R be a ring.

Recall (def): A map $\varphi: M \to N$ of R-modules is a gp hom that respects the R-actions, i.e., a map s.t.

(i) $\varphi(m_1 + m_2) = \varphi(m_1) + \varphi(m_2)$ $\quad \forall m_1, m_2 \in M$

(ii) $\varphi(r \cdot m) = r \cdot \varphi(m)$ $\quad \forall r \in R, m \in M.$

Remarks.

(1). "that recurring theme". If $R = A$ is a k-algebra, then an A-module hom is automatically a linear map, that is, in addition to (i) it respects scaling: $\varphi(c\cdot m) = c\cdot\varphi(m)$ $\forall c \in k$, $m \in M$. Pf: Ex. Use $k \to A$, $c \mapsto c\cdot 1_A$.

(2) To check a map $\varphi: M \to N$ between two R-modules is a module hom, it suffices to check that φ is a gp hom and respects the R-actions.

feel free to quote results from gp theory.

e.g. (Inclusion & projection) M an R-module. $U \subseteq M$ a submodule.

The maps $\iota: U \to M$, $u \mapsto u$; $\pi: M \to M/U$, $m \mapsto m + U$

are both R-module homs.

$$\begin{tikzcd} m \arrow[r, "r."] \arrow[dr, "\pi"'] & r.m \arrow[r, "\pi"] & r.m+U \arrow[d, equal, "\text{def of } M/U"] \\ & m+U \arrow[r, "r."] & r.(m+U) \end{tikzcd}$$

Check: gp hom? ι ✓ π ✓ by gp theory; respect action? easy to check as well.

Isomorphism Thms for modules

Thm. (Thm. 2.24). Let R be a ring, M, N be R-modules, and $U, V \subseteq M$ be submodules of M.

(a) If $\varphi: M \to N$ is an R-module hom. Then $\operatorname{Im}\varphi \subseteq N$ is a submodule of N, $\ker\varphi$ is a submodule of M, and we have a well-defined module iso

$$M/\ker\varphi \longrightarrow \operatorname{Im}\varphi, \qquad m + \ker\varphi \longmapsto \varphi(m).$$

(b) The sum $U+V$ is an R-module, the intersection $U \cap V$ is an R-module, and we have $\quad U/_{U\cap V} \cong {(U+V)}/_{V}$.

$$\begin{tikzcd}[column sep=small] & U+V \arrow[dl, dash, dashed] \arrow[dr, dash, dashed] & \\ U \arrow[dr, dash, dashed] & & V \arrow[dl, dash, dashed] \\ & U\cap V & \end{tikzcd}$$

(c) Suppose $U \subseteq V$. Then V/U is an R-submodule of M/U, and we have $\dfrac{(M/U)}{(V/U)} \cong M/V$ as R-modules.

Pf: Hw.

Internal/external direct sums.

In HW4, you are asked to show the last claim from Lecture 10.

M: R-module. $(U_i)_{i\in I}$, each U_i a submod of M.

external ↙ ↘ internal way

Can create the ext. dir. sum $\overset{E.}{\bigoplus} U_i$

there's a chance that M equals the internal direct sum $\overset{I.}{\bigoplus} U_i$, i.e. that $(U_i)_{i\in I}$ satisfy the conditions in Def 2.15 (b)

Claim: If so, $\overset{E.}{\bigoplus} U_i \longrightarrow G$, $(u_i)_{i\in I} \longmapsto \sum_{i\in I} u_i$ gives an R-mod iso.

Preservation of scaling actions.

Prop. Let A be a k-alg. and $\varphi : M \to N$ an A-module hom. Suppose that some elt $r \in A$ acts as a scalar on a certain elt $m \in M$, say $r \cdot m = cm$ for some $c \in k$. Then r must also act as the same scalar on $\varphi(m)$, i.e., $r \cdot \varphi(m) = c\,\varphi(m)$.

Pf: $r \cdot \varphi(m) = \varphi(r \cdot m) = \varphi(cm) = c\,\varphi(m)$.

The simple prop is quite useful for distinguishing non isomorphic A-modules: given M, N modules, if some $r \in A$ acts as a scalar c on some $m \in M$ but acts as c on no elt of N, then $M \not\cong N$.

application:

reason for fact: next time

Fact: Let $A = \mathbb{C}[x]/(x^n-1)$ and $U = \{ x \in \mathbb{C} : x^n = 1 \}$.

For all $\lambda \in U$, there is a simple 1-dim A-mod V_λ

s.t. $V_\lambda = \mathbb{C}$ as v.s. and $\underbrace{\bar{x} \cdot 1 = \lambda \cdot 1}_{(*)}$. $\left(\bar{x} = x + \langle x^n - 1 \rangle \right)$

Note: · $(*)$ implies that $\bar{x}$ acts as λ on all of V_λ:

$$\forall c \in V_\lambda = \mathbb{C}, \quad \bar{x} \cdot c = \bar{x} \cdot (c1) = c(x \cdot 1) = c\lambda = \lambda c.$$

· By the prop, since $\bar{x}$ acts on all of V_λ as λ and on all of V_μ as μ for all $\mu, \lambda \in U$, so if $\mu \neq \lambda$ then $V_\lambda \not\cong V_\mu$.