Math 4140. Lecture 10.

02.08.2021

Last time:

- Submodules
- Simplicity of $M_n(k) \circlearrowright k^n$
- quotient modules.

Today.

- Definition of module homomorphisms
- More new modules from old: "induced" modules, direct sums / products.

1. Def. of module homomorphisms

Def. Let R be a ring and let M, N be R-module. A map $\phi : M \to N$ is called an R-module homomorphism if

i) $\phi(m_1 + m_2) = \phi(m_1) + \phi(m_2) \quad \forall m_1, m_2 \in M$

ii) $\phi(r \cdot m) = r \cdot \phi(m) \quad \forall r \in R, m \in M.$

An R-module hom $\phi : M \to N$ is called an isomorphism if it's bijective.

Note: We'll look at module hom more carefully soon.

Right now, we'll just note that there's a natural notion of "isomorphic modules".

2. "Induced" modules (Example 2.4. (4) & (5))

Let A, B be rings and let M be an B-module.

(a) If $A \subseteq B$, ie., if A is a subring of B, then M is naturally also an A-module with the same (inherited) action. **Ex:** Convince yourself that the axioms hold.

$$\underset{A \subseteq B}{\hat{a}} \cdot m = \underline{a \cdot m} \quad \forall a \in A, m \in M.$$

↳ whatever it is in M as a B-module

Eg. $k \cong \underline{\underline{A}} = \{\lambda \mathrm{Id}_n \mid \lambda \in k\} \subseteq \underline{\underline{B}} = M_n(k) \circlearrowright k^n =: \underline{\underline{M}}$, $n > 1$.

Jacob's question: M is simple as a B-module. Is it still simple as an A-module?

Answer: No. For all $1 \leq i \leq n$, $M_i := \mathrm{Span}(e_i)$ is a submodule of M when M is viewed as an A-module. In fact, $\mathrm{Span}(v)$ is a submodule $\forall v \in k^n$.

(b) More generally, if there is a ring hom $\varphi : A \to B$

(for (1), $\varphi = i : A \to B$, $i(a) = a \ \forall a \in A$)

then any B-module M can be made an A-module with the action given by

$$a \cdot m = \varphi(a) \cdot m \qquad \forall a \in A, m \in M$$

Let's check the axioms.

(i) $a \cdot (m+n) \boxed{=} \underbrace{\varphi(a)}_{B} \cdot (m+n) = \varphi(a) \cdot m + \varphi(a) \cdot n \boxed{=} a \cdot m + a \cdot n$

(ii) $1_A \cdot m \boxed{=} \varphi(1_A) \cdot m = 1_B \cdot m = m$

(iii) $(a+a') \cdot m \boxed{=} \varphi(a+a') \cdot m = [\varphi(a) + \varphi(a')] \cdot m = \varphi(a) \cdot m + \varphi(a') \cdot m \boxed{=} a \cdot m + a' \cdot m$

(iv) $(aa') \cdot m \boxed{=} \varphi(aa') \cdot m = [\varphi(a)\varphi(a')] \cdot m = \varphi(a)[\varphi(a') \cdot m] \boxed{=} \varphi(a)[a' \cdot m] \boxed{=} a \cdot (a' \cdot m)$.

$\forall a, a' \in A$, $m, n \in M$

$\bigcirc$: φ is a hom

$\square$: def of $A \circlearrowright M$

Note: We'll have another explanation for why M is "certainly" an A-module this way.

3. Direct products / sums

Def. (External direct sum / product ; Def 2.17, P.38)

Let R be a ring and let $(M_i)_{i\in I}$ be a family of R-modules for some index set I.

(a) The Cartesian product $\prod_{i\in I} M_i = \{ (m_i)_{i\in I} \mid m_i \in M_i \text{ for all } i\in I \}$

is naturally an R-module with componentwise add. an R-action. ($(m_i)_i + (n_i)_i = (m_i + n_i)_i$; $r\cdot (m_i)_i = (r\cdot m_i)_i$) ; it is called the <u>direct product</u> of the family $(M_i)_{i\in I}$.

(b) The subset $\bigoplus_{i\in I} M_i = \{ (m_i)_{i\in I} \mid m_i \in M_i \ \forall i, \ m_i \neq 0 \text{ for only finitely many } i\in I \}$.

of $\prod_{i\in I} M_i$ is a submodule of $\prod_{i\in I} M_i$; it's called the <u>direct sum</u> of $(M_i)_{i\in I}$.

Note: If the indexing family I is finite, then $\prod M_i = \bigoplus M_i$.

Notation:

$[n] := \{1, 2, 3, \cdots, n\}$

$\forall\, n \in \mathbb{Z}_{\geq 1}$.

Def (internal direct sum; Def. 2.15.(b). P37)

Let R be a ring and let M be an R-module.

We say M is the direct sum of a family $(U_i)_{i\in I}$ of R-submodules, denoted

$M = \bigoplus_{i\in I} U_i$, if the following holds:

(i). $M = \sum_{i\in I} U_i$, that is, every elt $m \in M$ is a finite sum of elts from the submodules U_i.

(ii). For every $j \in I$ we have $U_j \cap \sum_{i\neq j} U_i = 0$

E.g. We saw that for $k \cong \underline{\underline{A}} = \{\lambda\, Id_n \mid \lambda \in k\} \subseteq \underline{\underline{B}} = M_n(k) \curvearrowright k^n =: \underline{\underline{M}}$ $(A \curvearrowright M)$

$U_i := \mathrm{Span}(e_i)$ is a submodule of M for all $1 \leq i \leq n$. The collection $(U_i)_{i\in[n]}$ satisfies (i) and (ii) by linear algebra, so $M = \bigoplus_{i=1}^{n} U_i$.

Note: (1) Recall from gp theory that for a gp G and a family $\{G_i : i \in I\}$ of subgps of G, G is isomorphic to the external direct sum $\overset{Ex}{\bigoplus_{i\in I}} G_i$ iff $G = \bigoplus_{i \in I} G_i$ as an internal direct sum.

via the map $\overset{E.x.}{\bigoplus} G_i \longrightarrow G$, $(g_i)_{i\in I} \longmapsto \sum g_i$.

(2) The same fact holds for R-modules: for an R-module M and a family of submodules $(U_i)_{i\in I}$. M is iso to the external dir. sum $\overset{E.x.}{\bigoplus} U_i$ iff $M = \bigoplus U_i$ as an internal direct sum (i.e., iff (i) and (ii) hold) via the map. $\overset{E.x.}{\bigoplus} U_i \longrightarrow \bigoplus U_i$, $(u_i)_{i\in I} \longmapsto \sum_{i\in I} u_i$.

Pf: HW.