Math 4140. Lecture 9

02.05.2021.

Last time: – def and first def of modules over rings and algebras

M (modules), R (rings)

modules $\downarrow$ an abelian gp equipped with an action of the ring w/ nice properties.
$R \times M \to M$

" modules are to rings what v.s. are to fields (k-modules $\equiv$ k-v.s.) "

" $\mathbb{Z}$-modules $\equiv$ abelian gps "

regular actions. $\quad R \circlearrowright R$ (left mult. $r \cdot s = rs$) $\xrightarrow{\text{generalize}}$ $R \circlearrowright I$ (any ideal I of R)

natural actions. $\quad M_n(k) \circlearrowright k^n \quad \xleftrightarrow{\text{"same"}} \quad \mathrm{End}_k(V) \circlearrowright V.$

matrix-vec. mult. $A \cdot \vec{v} = A\vec{v}$

evaluation $f \cdot v = f(v)$

Today. New modules from old (very loosely speaking):

1. submodule , 2. quotient modules

3. "induced" modules. , 4. direct product / sum

1. Submodules

Def. (Def 2.12) Let R be a ring and M an R-module. A <u>submodule</u> of M is a subgp $U \subseteq M$ closed under the R-action, i.e., s.t.

$r \cdot u \in U \quad \forall r \in R, u \in U.$

$\Longrightarrow$ the module axioms must still hold "for u", e.g. $(rs) \cdot u = r \cdot (s \cdot u)$, $\forall u \in U \subseteq M$.

so M is an R-module.

Note. (1) As usual, if R is a k-algebra, then a submodule of M is automatically a subspace of M.

(2) The underlined condition is equivalent to the condition that $U \subseteq M$ is an R-module in its own right under the action inherited from M.

(See the green text on the last page.)

(3) (Normality) Since M is an abelian gp, all subgps of M are normal, including U.

Examples (a) $R = K$. an R-module $\equiv$ a k-v.s V

$\Downarrow$

a submodule of V $\equiv$ a subspace of V

(b) $M = R$. Submodules $\equiv$ (left) ideals of R.

$R \circlearrowright R$

(c). Simplicity of $M_n(k) \circlearrowright k^n =: V$. $k \to A$ $\lambda \mapsto \lambda 1_A$

Def. An R-module is <u>simple</u> if it is nonzero and it has no proper nontrivial submodules (i.e., its submodules are 0 and itself).

Prop. $V := k^n$ is a simple module of $A := M_n(k)$.

pf: Take a nontrivial submodule $W \leq V$. Then W has a nonzero vector $w = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, with $a_i \neq 0$ for some $1 \leq i \leq n$. It suffices to show that $W = V$. Since W is a subspace of V, it suffices to show that W contains the standard basis

$$\beta = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Now, say $a_j \neq 0$. Consider the matrix units E_{ij}, $1 \leq i \leq n$. Then $E_{ij} \cdot w = E_{ij} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \underset{\text{lin. alg.}}{\overset{*}{=}} a_j \cdot e_i$.

(e.g. $n=3$. $E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $E_{12} \cdot \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$)

It follows that W contains $\{a_j e_1, a_j e_2, \ldots, a_j e_n\}$ and hence β, as desired. $\square$

2. Quotient modules

Let R be a ring, let M be an R-module, and let $U \subseteq M$ be a submodule (so U is a normal subgp of M as we noted and M/U makes sense as a quotient gp).

Once we define module homs, we'll have a 1st iso thm for modules.

Prop (Def. 2.18) The quotient gp M/U is naturally an R-module under the action inherited from M, i.e., under the action

$$\varphi: R \times M/U \to M/U, \quad r\cdot(m+U) = (r\cdot m) + U$$

We call M/U <u>the quotient module of M by U</u>.

Pf: (1) φ is well-defined: $m_1 + U = m_2 + U \overset{\text{gp theory}}{\Rightarrow} m_1 - m_2 \in U$

$\overset{\text{"}U \text{ is a submodule"}}{\Longrightarrow} r\cdot(m_1 - m_2) \in U \Rightarrow r\cdot m_1 - r\cdot m_2 \in U \Rightarrow r\cdot m_1 + U = r\cdot m_2 + U$

(2). The action satisfies the necessary axioms: HW (inheritance).

Examples.

(1). $R = \mathbb{Z}$, $M = \mathbb{Z}$, $U = \langle d \rangle$.

The gp quotient $M/U = \mathbb{Z}/\langle d \rangle =: \mathbb{Z}/d\mathbb{Z}$ is familiar as a gp from gp theory. e.g. $d = 6$. $\to$ $M/U = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$.

The action: mult. followed by "mod. 6".

$$r = 10 \in R. \qquad m = 2 \qquad m + U = 2 + U$$

$$r \cdot (m + U) = 10 \cdot \bar{2} = \underset{(18+2)}{20 + U} = \bar{2}.$$

(2). $Q = (Q_0, Q_1)$ a quiver. $Q_1 \subseteq$ $e_a\ (a \in Q_0)$

$R = kQ$, $M = kQ$, $U = kQ^{\geq 1} := \mathrm{Span}\{$all paths on Q of length at least 1$\}$.

A basis for M/U: $\{e_a + U : a \in Q_0\}$.

Pf: HW.

$p \cdot (x + U) = 0$ for any path p of length ≥ 1.

The actions: mostly zero: everything in U acts as zero on M/U

What about the actions of the sta. paths?

$$e_a \cdot \underbrace{(e_b + U)}_{\text{typical basis elt of } M/U} = \delta_{ab}(e_b + U)$$

$\boxed{\text{More examples of module constructions next time.}}$