Math 4140. Lecture 8. 02.03.2021.

Last time:

- Factor algebras.
- The first isomorphism thm for algebras
- $k[x]$ is a PID. → more on the next two pages

Today. We start Chapter 2.

- Modules of rings and algebras: def. and examples

(Algebras are rings, so we'll define modules over rings. The generalization causes no complications for us.)

Facts on $k[x]$ (to be used later)

(1) $k[x]$ is a PID

Explanation: The fact that $k[x]$ is a PID can be deduced from the fact that it is a Euclidean domain, that is, it has a division algorithm with certain properties: $a, b \rightsquigarrow a = qb + r$

If you haven't seen this, consult other standard algebra books for details.

E.g. $a = x^3 - 6x^2 + x$

$b = x^2 + 2x$

$$a = x(x^2+2x) - 8x^2 + x$$
$$= x(x^2+2x) - 8(x^2+2x) + 17x$$
$$= \underbrace{(x-8)}_{q}\underbrace{(x^2+2x)}_{b} + \underbrace{17x}_{r}$$

E.g. The ring $(\mathbb{Z}, +, \cdot)$ is a Euclidean domain under the usual division algorithm. Every ideal I in $\mathbb{Z}$ can be written as $I = \langle n \rangle$ where n is the smallest positive number in I.

Similarly, every ideal $I \subseteq k[x]$ can be generated by the monic polynomial in I with the smallest degree.

(2) Quotients of $k[x]$

Prop. Let $A = k[x]$ and let I be an ideal in A.

Then A/I is fin. dim.

Pf: Say $I = \langle f \rangle$ and $\deg f = d$. Then

$\{1, x, x^2, \ldots, x^{d-1}\}$ is a basis for A/I.

Details: HW / Eg. 1.21

Modules over rings. ← Ch. 2.

Recall that a v.s. over a field K is an abelian gp $(V,+)$ with an action (scalar multiplication) of K $K\times V\to V$ satisfying certain properties/axioms.

Replacing the field K with a ring R in the v.s. axioms "the same" formally. exactly

we'll almost always talk about left modules only.

Def: (EH. Def 2.1). A (left) module over a ring R is an abelian gp $(M,+)$ with an action of R, i.e., with a map $R\times M\to M$ satisfying the axioms that $\forall r,s\in R,\ m,n\in M$,

"hom"-like properties

(a) $r\cdot(m+n)=r\cdot m+r\cdot n$ (the action of r respects $+$)

(b) $(r+s)\cdot m=r\cdot m+s\cdot m$ (action of $r+s$ = the sum of the actions of r and s)

(c) $r\cdot(s\cdot m)=(rs)\cdot m$ (action of s ∘ action of r = action of rs)

(d) $1_R\cdot m=m$ (action of the ring id = the id. action).

"that recurring theme": (see P1. of Lec7.pdf)

Prop: (Lemma 2.5) Let M be an R-module for a ring R.

If R is a k-algebra, then M is a k-vector space.

Pf: HW. Same key idea as before: think of k as embedded in

$A := R$ via $\lambda \mapsto \lambda \cdot 1_A$, then try to show that $\lambda m \in M$ $\forall m \in M$.

$\square$

Examples.

Let R be a ring.

(1). Vector space. If R is a field $R = k$, then the R-module axioms coincide with k-vec. space axioms, so R-modules coincide with k-vec. spaces.

(2). $\mathbb{Z}$-modules. If $R = \mathbb{Z}$, then an R-module is certainly an abelian gp by def. On the other hand, given an abelian gp $(\overset{M}{\underset{G}{\shortparallel}}, +)$, then we may define

$$n \cdot a = \begin{cases} 0 & \text{if } n = 0 \\ \underbrace{a + a + a + \cdots + a}_{n \text{ copies}} & \text{if } n > 0 \\ -(\underbrace{-n}_{>0} \cdot (a)) & \text{if } n < 0. \end{cases} \qquad \forall a \in M$$

In this sense, $\mathbb{Z}$-modules are the same as abelian gps.

So module theory encompasses ab. gp theory.

Easy exercise: the action thus defined makes M a $\mathbb{Z}$-module.

(3). regular modules. The ring R is a left module of itself, with the R-action being left mult: $R \times \underset{R}{\underset{\shortparallel}{M}} \to R$ $\quad a \cdot m = \underbrace{am}_{\text{mult. in } R}$.

What about the axioms? (0) ab gp? Yes, R is one. (a) "$a\cdot(m+n) = a\cdot m + a\cdot n$"

(b) "$(a+b)\cdot m = a\cdot m + b\cdot m$" (c) $a\cdot(b\cdot m) = (ab)\cdot m$ (d) $1_R \cdot m = m$

(a), (b): distributivity (c) associativity (d) unit axiom for R.

(4) ideals. Any left ideal $I \subseteq R$ is a left module (again with left mult in R being the action).

(0) abelian gp? ✓ by def (a) (b) (c) (d) $\to$ same laws in R as in the previous example plus "closure": $R \cdot I \subseteq I$.

(15). Direct products: new modules from old.

We'll skip this for now. See E.g. 2.3.(15).

Now we consider some examples where the ring is an algebra.

(6). natural modules of matrix algebras

(i). The algebra $A = M_n(k)$. $\leftarrow R$ The v.s. $V = k^n = \left\{ \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} : a_i \in k \ \forall i \right\}$

is an A-module

The action: $A \times V \to V$ matrix-vector multiplication

e.g. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

The axioms: ab. gp ✓. (a) – (d): properties of matrix-vec. mult.

(ii). More generally, any subalgebra of $M_n(k)$ also has $V = k^n$ as a left module, with action being multiplication.

(7). natural modules of endomorphism algebras $\mathrm{End}_k(V) \curvearrowright V$

The action : evaluation

$$\begin{array}{ccc} \mathrm{End}_k(V) \times V & \longrightarrow & V \\ \rotatebox{90}{$\in$} & & \\ (f, v) & \longmapsto & f(v). \end{array}$$

The axioms : HW.

Note: This should be no surprise : we saw $\mathrm{End}_k(V) \cong M_n(k)$ for $n = \dim V$. Can you describe the connection between E.g. (7) and E.g. (6) more precisely ?

"identify"

More examples next time.