Math 4140. Lecture 7.

02.01.2021

Last time:

- Examples of path algebra (algebra $\longleftrightarrow$ combinatorics)
- Ideals of rings and algebras.

A theme that will recur:

- For a ring R, a left ideal $I \subseteq R$ is defined as a ~~subgp~~ subgp of R closed under left mult. by R.
- If R is an K-algebra $R = A$, then the underlined conditions automatically guarantees that I is a subspace.

  a relevant fact: K embeds into any K-algebra A via the map $\lambda \longmapsto \lambda \cdot 1_A$

So, it's equivalent to define ~~define~~ a left ideal of R as a subspace closed under left mult. by R.

Today · More on factor/quotient algebras

· The first iso. thm. for algebras

· Ideals in polynomial rings. (preparation for ch 2)

Factor algebras

We stated: If I is a two-sided ideal of A (hence I is a subspace),

then A/I is an algebra.

Let's sketch the proof more carefully (assuming no ring theory): We need:

(1). A/I is a vector space, with cosets $\{a+I : a \in A\}$ as elts, addition and scaling given by $(a+I)+(a'+I)=(a+a')+I$, $\lambda \cdot (a+I) = \lambda a + I$.

linear algebra recollection

exercise

12). A / I has a well-defined mult : $(a+I)(b+I) = ab+I$.

needs closure of I under mult A on both sides.

Well-definedness. if $a+I = a'+I$, $b+I = b'+I$, then $ab+I = a'b'+I$.

Pf: $a+I = a'+I$, $b+I = b'+I \Rightarrow a-a' \in I$, $b-b' \in I$

$\Rightarrow ab - a'b' = (ab-ab') + (ab'-a'b') = a\underbrace{(b-b')}_{I} + \underbrace{(a-a')}_{I}b \in I \Rightarrow ab+I = a'b'+I$.

since I is a two-sided ideal.

13). The mult. in A/I is unital, associative, bilinear, and distributes over addition.

Pf: follows from the def of the operations and inheritance of the corresponding properties.

e.g. assoc. of mult in $A/I \Leftarrow$ assoc. of mult. in A.

The first isomorphism theorem.

Thm. Let $\varphi : A \to B$ be a hom. of algebras. Then

(Read Thm 1.26)

(1). The <u>image</u> $\mathrm{Im}\,\varphi := \{\varphi(a) : a \in A\}$ is a subalgebra of B.

(2). The <u>kernel</u> $\ker\varphi := \{a \in A : \varphi(a) = 0\}$ is a two sided ideal of A.

(3). We have a well-defined algebra isomorphism

$$\bar{\varphi} : A/\ker\varphi \to \mathrm{Im}\,\varphi \quad , \quad a + \ker\varphi \mapsto \varphi(a).$$

<u>Pf sketch</u>: (1). Routine. <u>One way</u>: (i). $\mathrm{Im}\,\varphi$ is a subspace of B : linear algebra. (ii). closure under mult: $\varphi(a)\varphi(b) = \varphi(ab) \in \mathrm{Im}\,\varphi$.

(iii). $1_B \in \mathrm{Im}\,\varphi$: $\varphi(1_A) = 1_B$.

(2). $\ker\varphi$ is an ideal: (i). $\ker\varphi$ is a subgp: gp theory.

(ii). closure under left mult: $\forall a \in A$, $k \in \ker\varphi$, $\varphi(ak) = \varphi(a)\underbrace{\varphi(k)}_{\substack{\shortparallel \\ 0}} = 0$;

(iii) . - - - right mult: similar.

algebra	=	vect space	=	abelian gp
				scaling
		mult		mult

(3). $\bar{\varphi}: A/\ker\varphi \to \mathrm{Im}\,\varphi$ is a well-defined algebra iso:

(i). By gp theory, $\bar{\varphi}$ is well-defined, a gp hom, and bijective.

(first iso thm for gps)

So it remains to show that (ii) $\bar{\varphi}$ respects scaling:

$$\bar{\varphi}(\lambda\cdot(a+\ker\varphi)) = \bar{\varphi}(\lambda a + \ker\varphi) = \varphi(\lambda a) = \lambda\varphi(a) = \lambda\bar{\varphi}(a+\ker\varphi)$$

and (iii) $\bar{\varphi}$ respects mult:

$$\bar{\varphi}((a+\ker\varphi)\cdot(b+\ker\varphi)) = \bar{\varphi}(ab+\ker\varphi) = \varphi(ab) = \varphi(a)\varphi(b) = \bar{\varphi}(a+\ker\varphi)\,\bar{\varphi}(b+\ker\varphi).$$

$\square$

Example. ($\mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$ as $\mathbb{R}$-algebras.)

- (Evaluation homs.) Let A be a K-algebra. For every $a \in A$ the evaluation map $\mathrm{Eval}_a : K[x] \to A$, $\sum \lambda_j x^j \mapsto \sum \lambda_j a^j$ is an algebra hom.

Pf: HW.

Eval$_a$ is surj. so $\mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$.

- Take $K = \mathbb{R}$, $A = \mathbb{R}[x]$, $B = \mathbb{C}$, and $a = i \in \mathbb{C}$. Then the kernel of the evaluation map $\mathrm{Eval}_a : A \to B$, i.e., $\mathrm{Eval}_i : \mathbb{R}[x] \to \mathbb{C}$ is given by $\ker \mathrm{Eval}_i = \{ f \in \mathbb{R}[x] : \mathrm{Eval}_i(f) = 0 \} \overset{HW}{\underset{Ex 1.27.(1)}{=}} (x^2+1)$.

"$\supseteq$" ✓

"$\subseteq$"

Meanwhile, in B we have $1 = \mathrm{Eval}_a(1) = \mathrm{Eval}_a(x^0)$ "$f(a)$" and $i = \mathrm{Eval}_a(x)$, so $\{1, i\} \subseteq \mathrm{Im}\,\mathrm{Eval}_a$, so $B \subseteq \mathrm{Im}\,\mathrm{Eval}_a$.

Ideals in $k[x]$:

Fact: Every ideal in $k[x]$ is principal, i.e., every ideal I is generated by a single elt $f \in k[x]$. (so $k[x]$ is a <u>principal ideal domain</u> or <u>PID</u>.)

Note: $k[x,y]$ is not a PID: the ideal $I = \langle x, y \rangle$ cannot be generated by a single elt.

Pf: HW.

Similarly, for any $n \geq 2$, $k[x_1, x_2, \ldots, x_n]$ is not a PID.

More next time.