

Math 4140. Lecture 6.

Last time: Monoids (generalization of gps)

• Monoid algebras $M \rightarrow \underline{KM}$: basis: M , mult on basis = concat,
↓ extend bilinearly
mult in general

if M is a
gp G
1) gp algebra $G \rightarrow kG$.

2) free algebra of a set X : monoid algebra
of the free monoid $\langle X \rangle$ of X .

$$X \rightarrow M = \langle X \rangle \rightarrow \underline{KM} = k\langle X \rangle.$$

• Path algebras of quivers. $Q \rightarrow \underline{kQ}$: basis = {paths on Q }, mult: induced from concatenation.

Today.

- Examples of path algebras.
- Ideals and quotient/factor algebras.

Examples of path algebras

given an arrow $a \xrightarrow{\alpha} b$,
 $\alpha e_a = \alpha = e_b \alpha$

(1) $1 \xleftarrow{\alpha} 2$

Question: What's $\dim kQ$?

↑ a relation in every path algebra

$$\dim kQ = |\text{any basis of } Q| = \# \text{ paths on } Q.$$

$$= \{ \alpha, e_2, e_1 \} \quad \alpha \cdot e_1 = 0 \quad e_1 \cdot \alpha = \alpha$$

(2) $\begin{array}{c} \alpha \\ \curvearrowright \\ \downarrow \end{array}$

Question: $kQ \cong k[x]$? Proof?

→ Pf: HW.

"one loop" quiver

Yes. The linear map with $\alpha^i \mapsto x^i \forall i \geq 0$ is an alg. iso.

basis for kQ : $\{ e_v, \alpha, \alpha^2 := \alpha \cdot \alpha, \alpha^3, \dots \}$. $\therefore kQ$ is inf. dim.

eg. mult:

$$(2\alpha - \alpha^2) (e_v + 3\alpha^3) = 2\alpha - \alpha^2 + 6\alpha^4 - 3\alpha^5$$

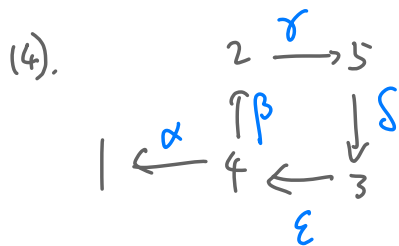


Question: $kQ \cong k[x, y]$? **No**, $k[x, y]$ is comm., kQ is not.

basis: $e, x, y, xx, xy, yx, yy, \dots$ kQ is inf. dim

Prop. $kQ \cong k\langle a, b \rangle$ (i.e., $k\langle X \rangle$ where $X = \{a, b\}$)

Pf. Hw. Use univ. prop. of $k\langle x, y \rangle$ to find an algebra hom from $k\langle a, b \rangle$ to



Question: Is kQ f.d. ? **No**

For what Q is kQ f.d. ?

kQ , then check if's bijective.

(say $|Q_0| < \infty$)

"if Q has no directly cycles".

$\delta\alpha\beta = \delta\sigma \cdot \beta$

basis for kQ :

- e_i ($1 \leq i \leq 5$), $\alpha, \beta, \sigma, \delta, \epsilon,$
- $\sigma\beta, \delta\sigma, \epsilon\delta, \alpha\epsilon, \beta\epsilon,$

$\dots, \left(\begin{matrix} \epsilon\delta\sigma\beta \\ \epsilon\delta\sigma\beta \end{matrix} \right)^j$ ($j \geq 1$), \dots

Pf. Hw.

1. Subalgebras, ideals and factor/quotient algebras.

$(A, +, \cdot, \iota, m)$
||

Recall that we've defined a subalgebra of an algebra A ($\ni 1_A$) to be a subspace $B \subseteq A$ that contains 1_A and is closed under mult.

Def (Def 1.17). A left ideal in an algebra A is a subgp I of A closed under left mult. by A in the sense that \rightarrow right ideals are defined similarly.

$$a \cdot x \in I \quad \forall a \in A, x \in I \quad (\text{ie. } A \cdot I \subseteq I)$$

A two-sided ideal is a subgp $I \subseteq A$ st. $a \cdot x \cdot b \in I \quad \forall a, b \in A, x \in I$.

Note 1: The definition of an ideal I does not require $1_A \in I$.

applies to rings as well.

Indeed, if $1_A \in I$, I a left ideal, then $a \cdot 1_A \in I \quad \forall a \in A$, ie. $a \in I \quad \forall a \in A$.
If A is commutative, then the three notions of ideals coincide, ie. $A = I$

Prop. (Lemm 1.20.) Let A be a K -algebra. Then

(1) Every left ideal $I \subseteq A$ is a K -subspace of A .

(2) If I is a proper two-sided ideal of A , then the factor ring A/I is a K -algebra (it's called the quotient/factor algebra of A w.r.t. I).

Why? " K embeds into A via $\lambda \mapsto \lambda \cdot 1_A$ "

→

Pf (sketch): (1). By def, I is a subgroup and hence closed under addition, so

it suffices to show it's closed under scalar mult: $\forall \lambda \in K, x \in I$.

$$\lambda \cdot x \stackrel{\text{def of } 1_A}{=} \lambda (1_A \cdot x) \stackrel{\text{b.l.}}{=} \underbrace{(\lambda 1_A)}_{\substack{\uparrow \\ \text{def of ideal}}} \cdot x \in I. \quad \checkmark$$

(2). HW/next time (recall that mult in A/I is given by $(a+I)(b+I) = ab + I$.)

Examples of ideals and factor algebras

Let A be an algebra

- (One-sided Principal ideals, E.g. 1.19. (2).) If $x \in A$.

$(x)_L := \{ a \cdot x \mid a \in A \}$ is a left ideal; it's called the principal left ideal generated by x and sometimes just denoted Ax .

- In the path algebra kQ of a quiver $Q = (Q_0, Q_1)$, for any stationary path e_a ($a \in Q_0$), the left principal ideal $kQ \cdot e_a$

can be described as the span of all paths starting at a .

More
examples
next time.