Math 4140. Lecture 6.

Last time: · Monoids (generalization of gps)

· Monoid algebras $M \to \underline{kM}$: basis: M, mult on basis = concat, ↓ extend bilinearly mult in general

if M is a gp G

(1) gp algebra $G \to kG$.

(2) free algebra of a set X: monoid algebra of the free monoid $\langle X \rangle$ of X.

$X \longrightarrow M = \langle X \rangle \longrightarrow kM = k\langle X \rangle$.

· Path algebras of quivers. $Q \to \underline{kQ}$: basis = {paths on Q}, mult: induced from concatenation.

Today. · Examples of path algebras.

· Ideals and quotient/factor algebras.

Examples of path algebras

given an arrow $a \xrightarrow{\alpha} b$,

$$\alpha e_a = \alpha = e_b \alpha$$

↗ a relation in every path algebra

(1) $1 \xleftarrow{\alpha} 2$

<u>Question</u>: What's $\dim kQ$?

$$\dim kQ = |\text{any basis of } Q| = \#\text{ paths on } Q.$$

$$= \{\alpha, e_2, e_1\} \qquad \alpha \cdot e_1 = 0 \quad e_1 \cdot \alpha = \alpha$$

(2)
$$\begin{tikzcd} v \arrow[loop above, "\alpha"] \end{tikzcd}$$

"one loop" quiver

<u>Question</u>: $kQ \cong k[x]$? Proof ? → Pf: HW.

Yes. The linear map with $\alpha^i \mapsto x^i$ $\forall\, i \geq 0$ is an alg. iso.

basis for kQ: $\{ e_v, \alpha, \alpha^2 := \alpha \cdot \alpha, \alpha^3, \ldots \}$. So kQ is inf. dim.

"α^0"

e.g. mult:
$$(2\alpha - \alpha^2)(e_v + 3\alpha^3) = 2\alpha - \alpha^2 + 6\alpha^4 - 3\alpha^5.$$

(13). $\begin{tikzcd} v \arrow[loop left, "x"] \arrow[loop right, "y"] \end{tikzcd}$

<u>Question</u>: $kQ \cong k[x,y]$? No, $k[x,y]$ is comm.; kQ is not.

<u>basis</u>: $e_v,\ x,\ y,\ xx,\ xy,\ yx,\ yy,\ \ldots\ldots$

$xy \neq yx$

kQ is inf. dim

<u>Prop.</u> $kQ \cong k\langle a,b\rangle$ (i.e., $k\langle X\rangle$ where $X=\{a,b\}$)

Pf: HW. Use univ. prop. of $k\langle x,y\rangle$ to find an algebra hom from $k\langle a,b\rangle$ to kQ, then check it's bijective.

(14). $\begin{tikzcd} & 2 \arrow[r, "\gamma"] & 5 \arrow[d, "\delta"] \\ 1 & 4 \arrow[l, "\alpha"'] \arrow[u, "\beta"] & 3 \arrow[l, "\varepsilon"] \end{tikzcd}$

<u>Question</u>: Is kQ f.d. ? No

$\underline{\delta\gamma\beta} = \underline{\delta\gamma} \cdot \underline{\beta}$

For what Q is kQ f.d. ? (say $|Q_0|<\infty$)

"iff Q has no directed cycles".

Pf: HW.

<u>basis for kQ</u>:

e_i $(1 \le i \le 5)$, $\alpha, \beta, \gamma, \delta, \varepsilon$,

$\gamma\beta,\ \delta\gamma,\ \varepsilon\delta,\ \alpha\varepsilon,\ \beta\varepsilon,$

$\ldots\ldots,\ \boxed{(\varepsilon\delta\gamma\beta)^j \quad (j \ge 1)},\ \ldots\ldots$

1. Subalgebras, ideals and factor/quotient algebras.

$$\left(\begin{array}{c}A, +, \cdot_k, m\end{array}\right)$$
$$\parallel$$

- Recall that we've defined a subalgebra of an algebra A $(\ni 1_A)$ to be a subspace $B \subseteq A$ that contains 1_A and is closed under mult.

- Def (~Def 1.17). A left ideal in an algebra A is a subgp I of A closed under left mult. by A in the sense that → right ideals are defined similarly.

$$a \cdot x \in I \quad \forall a \in A,\ x \in I \quad (\text{i.e. } A \cdot I \subseteq I).$$

A two-sided ideal is a subgp $I \subseteq A$ s.t. $a \cdot x \cdot b \in I$ $\forall a, b \in A$, $x \in I$.

Note!: The definition of an ideal I does not require $1_A \in I$.
Indeed, if $1_A \in I$, I a left ideal, then $a \cdot 1_A \in I$ $\forall a \in A$, i.e. $a \in I$ $\forall a \in A$, i.e. $A = I$.

applies to rings as well.

- If A is commutative, then the three notions of ideals coincide.

Prop. (Lemm 1.20.) Let A be a K-algebra. Then

(1) Every left ideal $I \subseteq A$ is a K-subspace of A.

(2). If I is a proper two-sided ideal of A, then the factor ring A/I is a K-algebra (Why? "K embeds into A via $\lambda \longmapsto \lambda \cdot 1_A$") (it's called the quotient/factor algebra of A w.r.t. I).

Pf (sketch): (1). By def, I is a subgp and hence closed under addition, so it suffices to show it's closed under scalar mult: $\forall \lambda \in K,\ x \in I$.

$$\lambda \cdot x \overset{\text{def of } 1_A}{=\!=} \lambda(1_A \cdot x) \overset{\text{b.l.}}{=} \underbrace{(\lambda 1_A)}_{\overset{\uparrow}{a}} \cdot x \in I. \quad \checkmark$$

def of ideal.

(2). HW/next time (recall that mult in A/I is given by $(a+I)(b+I) = ab + I$.)

Examples of ideals and factor algebras

Let A be an algebra

- (One-sided Principal ideals. E.g. 1.19. (2)) $\forall x \in A$.

  $(x)_L := \{ a \cdot x \mid a \in A \}$ is a left ideal; it's called the principal left ideal generated by x and sometimes just denoted Ax.

- In the path algebra kQ of a quiver $Q = (Q_0, Q_1)$, for any stationary path e_a $(a \in Q_0)$, the left principal ideal $kQ \cdot e_a$ can be described as the span of all paths starting at a.

More examples next time.