Math 4140. Lecture 5.

important, see E.H. Eg. 1.24.(7)

Last time: polynomial algebras, endomorphism algebras ($\cong$ suitable matrix algebras).

Def: An <u>algebra isomorphism</u> is a bijective alg. hom.

Today.

- monoid and group algebras.
- quivers and their path algebras.

Monoid and gp algebras

Def: (Monoid) A monoid is a pair $(M, \cdot)$ where $\cdot$ is a binary operation on M that has an identity and is associative.

"closure"

$\forall a, b \in M, \ a \cdot b \in M$

$\exists e \in M$ s.t. $m \cdot e = m = e \cdot m \ \forall m \in M$

$\forall a, b, c \in M, \ (a \cdot b) \cdot c = a \cdot (b \cdot c)$

Note: Monoids generalize groups: a gp is just a monoid where every elt is invertible. **Q:** What about rings?

A: A ring is a triple $(R, +, \cdot)$ s.t. $(R, +)$ is an abelian gp, $(R, \cdot)$ .... (you complete it!)

Def. Let M be a monoid. The monoid algebra over a field k is the free vector space kM where multiplication is defined as concatenation on M and extended bilinearly.

E.g. $M = \langle X \rangle$ where $X = \{a, b\}$.

The monoid algebra is $A = kM$.

mult on M: e.g. $ab \cdot b = abb$.

mult. in general: $v_1 = ab - b \quad v_2 = 3a + 2b$.

$$\Rightarrow \quad v_1 \cdot v_2 = (ab - b)(3a + 2b) = 3aba - 3ba + 2abb - 2bb.$$

Def. If G is a gp (hence a monoid), we also call the monoid algebra kG of G the <u>group algebra</u> of G.

<u>e.g.</u> Take $G = C_3 = \langle g : g^3 = 1 \rangle = \{1, g, g^2\}$, the cyclic gp of order 3. Then kG has basis $\{1, g, g^2\}$ and thus dimension 3. Typical elts in kG look like

$$v = cg - dg^2 \quad \text{and} \quad w = e \cdot 1 - f \cdot g \quad \text{where } c, d, e, f \in k.$$

We have
$$\begin{aligned} v \cdot w &= (cg - dg^2) \cdot (e \cdot 1 - f \cdot g) \\ &= ce \cdot (g1) - de\,(g^2 \cdot 1) - cf\,(gg) + df\,(\underbrace{g^2 \cdot g}_{=1}) \\ &= ce \cdot g - de \cdot g^2 - cf\, g^2 + df \cdot 1 = df \cdot 1 + ce \cdot g - (de + cf)\, g^2 \end{aligned}$$

Example: Free algebras. $\underset{\text{set}}{X} \longrightarrow K\langle X\rangle$: $X \to \underset{M}{\langle X\rangle} \longrightarrow KM$

Def 1. Let X be a set. We define the <u>free monoid on X</u> to be the set $\langle X\rangle$ of <u>all words</u> on the alphabet X ; multiplication in $\langle X\rangle$ is given by concatenation/juxtaposition.

– include the empty word, which is the identity in $\langle X\rangle$.

<u>e.g.</u> $X = \{a, b\}$, $aba,\ bbaaa \in \langle X\rangle$.
$(aba)\cdot(bbaaa) = ababbaaa$

Def 2. Let X be a set and $M := \langle X\rangle$ be the free monoid on X.
We define <u>the free algebra on X</u> to be the monoid algebra $A = KM = K\langle X\rangle$.

E.g. $X = \{a, b\} \xrightarrow{\text{free monoid}} M = \langle X \rangle. \xrightarrow{\text{free v.s.}} A = kM$

A = lin. comb. of words in M.

$= \{\phi, a, b, aa, ab, ba, bb, aaa, \dots\}$.

infinite set of words

inf. dim. with M as a basis.

$\ni$ say $k = \mathbb{C}$, $2 \cdot ab + 3 \cdot bab - bb$

Example mult. in $k\langle X \rangle$:

$v_1 = ab - b \qquad v_2 = 3a + 2b$.

$\Rightarrow \quad v_1 \cdot v_2 = (ab - b)(3a + 2b) = 3aba - 3ba + 2abb - 2bb$

Note: $k\langle X \rangle$ is not commutative unless $|X| = 1$: if $X = \{a, b\}$ $(a \neq b)$, then ab and ba are diff. words in $\langle X \rangle$ and hence not equal in $\langle X \rangle$ or $k\langle X \rangle$.

Prop. (Universal property of free algebras on a set.)

Let X be a set and let $k\langle X\rangle$ be the free algebra on X, and let $i : X \to k\langle X\rangle$ be the map with $i(x) = x \ \forall x \in X$. Then for any algebra A and any function $\varphi : X \to A$, there is a unique alg. hom $\bar{\varphi} : k\langle X\rangle \to A$ s.t.

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, hook, "i"'] \arrow[dr, phantom, "\circ"] & A \\ k\langle X\rangle \arrow[ur, dotted, "\exists!\,\bar{\varphi}"'] & \end{tikzcd}.$$

pf: HW.

Aside: One can prove the above prop directly, or by putting together several universal properties ($X \xrightarrow{\text{free monoid}} M = \langle X \rangle \xrightarrow{\text{monoid algebra}} kM = k\langle X \rangle$), including the following: (universal prop of the monoid algebra of a monoid)

Prop* Let M be a monoid. For any k-alg. A and any monoid map $f: M \to A$ (i.e., any map w/ $f(m_1 m_2) = f(m_1 m_2)$ $\forall m_1, m_2 \in M$), there's a unique algebra map $\bar{f} : kM \longrightarrow A$ s.t.

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, hook] & A \\ kM \arrow[ur, dashed, "\bar{f}"'] & \end{tikzcd}$$

Q: Can you prove Prop* and use it to prove the prop. on the last page?

Path algebras.

Def. A <u>quiver</u> is a directed graph $Q = (Q_0, Q_1)$ where Q_0 is the vertex set and Q_1 the set of directed edges or <u>arrows</u>. For each arrow $\alpha : a \to b \in Q_1$ $(a, b \in Q_0)$, we say α has <u>source</u> a and <u>target</u> b, and we write $s(\alpha) = a$ and $t(\alpha) = b$.

Def. (paths / stationary paths) A <u>path</u> on a quiver Q is a sequence $p := \alpha_r \cdots \alpha_2 \alpha_1$ of arrows in Q s.t. $t(\alpha_i) = s(\alpha_{i+1})$ $\forall\, 1 \le i \le r-1$. We define the <u>source of p</u> to be $s(p) := s(\alpha_1)$ and the <u>target of p</u> to be $t(p) := t(\alpha_r)$.

Def. For each vertex $a \in Q_0$, we define a stationary path e_a at a:

it's the path that "stays at a", i.e., it has source a and target a but length 0. (while each arrow has length 1).

E.g. $c \xleftarrow{\beta} b \xleftarrow{\alpha} a$

e_a, e_b, e_c are stationary path

α, β are arrows and have length 1.

$\beta\alpha$ is a path, but $\alpha\beta$ is not.

αe_a, $e_b\alpha$, $\beta e_b \alpha e_a$ are paths.

Def. (Path algebra) Let $Q = (Q_0, Q_1)$ be a quiver. The path algebra is the free vector space kP where P is the set of all paths on Q. To define mult on kP, we define it on P first and extend bilinearly, and on P we define

$$p_1 \cdot p_2 = \begin{cases} p_1 p_2 & \text{if } t(p_2) = s(p_1) \\ 0 & \text{otherwise} \end{cases} \quad \forall p_1, p_2 \in P.$$

E.g. For $Q: c \xleftarrow{\beta} b \xleftarrow{\alpha} a$, $\beta \cdot \alpha = \beta\alpha$, $\beta \cdot \beta = 0$, $\alpha \cdot \beta = 0$, $\alpha \cdot e_a = \alpha$

Note: (unit of a path algebra) $1_{kP} = \sum_{a \in Q_0} e_a$.

Notation: We often simply denote the path algebra of Q by $\underline{\underline{kQ}}$.

Examples for next time. (feel free to think about the questions!)

(1) $1 \leftarrow 2$

<u>Question</u>: What's $\dim kQ$?

(2)
$$\begin{tikzcd} v \arrow[loop above] \end{tikzcd}$$

<u>Question</u>: $kQ \cong k[x]$? Proof ?

(3).
$$\begin{tikzcd} v \arrow[loop left, "x"] \arrow[loop right, "y"] \end{tikzcd}$$

<u>Question</u>: $kQ \cong k[x,y]$?

(4).
$$\begin{tikzcd} & 2 \arrow[r] & 5 \arrow[d] \\ 1 & 4 \arrow[l] \arrow[u] & 3 \arrow[l] \end{tikzcd}$$

<u>Question</u>: Is kQ f.d. ?

For what Q is kQ f.d. ?