# Math 4140. Lecture 4.

01.25.2021

**Last time:**

- Def. of (unital!) algebras (over a field $k$)
  - $k$-vector space with an associative, bilinear and unital mult.
- Def of dim., commutativity, and algebra homomorphisms.
- First examples of algebras.

  $k/k$, $\mathbb{C}/\mathbb{R}$, $M_n(k)/k$.

**Today.**

- Def of subalgebras.
- More examples of algebras: polynomial and endomorphism algebras.

# 1. Subalgebras

**Def.** Let $A$ be a $K$-algebra with unit $1_A$. A subalgebra $B$ of $A$ is a subset of $A$ with $1_A \in B$ that is closed under linear combinations and multiplication.

"subspace" "subring"

(Def. 1.14). Equivalently (HW), a subalgebra $B$ of $A$ is a subset $B \subseteq A$ containing $1_A$ that is an algebra in its own right under the ops $+, \cdot, \times$ inherited from $A$.

See E.H. Def 1.14 & Ex. 1.3.

## 2. More examples of algebras.

(4). (polynomial algebras $K[x]$, $K[x,y]$, ...)

$A = K[x_1, x_2, \ldots, x_n]$ poly. over $n$ variables with coeff. from $K$.

v.s. ✓ asso., bilinear, unital mult? ✓, unit $= 1$

$\dim_K K[x] = \infty$ A basis: $\{1, x, x^2, x^3, \ldots\}$

A is commutative by def. $x_i x_j = x_j x_i$

see next page

$\boxed{\dim ?}$

comm? $\boxed{X}$

(5). (Endomorphism algebras $\mathrm{End}_K(V)$, $V$ a $K$-v.s.)

$\mathrm{End}_K(V) = \{\text{lin. maps } f: V \to V \ / K\}$

comp is not comm.

unit $= \mathrm{id}_V$

v.s. ✓ (check this!) mult. $\overset{=\text{composition}}{fg} = V \xleftarrow{g} V \xleftarrow{f} V$ has the necessary properties ✓

**E.x.** An important isomorphism: $\mathrm{End}_k(V) \cong M_n(k)$

Recall that given any $k$-vector space $V$ and $W$, say with $\dim V = n$ and $\dim W = m$, there is a set-wise bijection

$$\{\text{linear maps } T: \underset{n}{V} \to \underset{m}{W}\} \xleftrightarrow{\text{choose any basis}} M_{m\times n}(k) = \{\underset{\text{rows}}{m} \times \underset{\text{cols}}{n} \text{ matrices over } k\}.$$

$\beta = \{v_1, \dots, v_n\}$ of $V$ and basis $\gamma = \{w_1, \dots, w_m\}$ of $W$

(e.g. $V = \mathbb{R}^n$, $W = \mathbb{R}^m$ / $k = \mathbb{R}$)

$$T \longmapsto \underbrace{[T]_\beta^\gamma = \left[\, [T(v_1)]_\gamma \;\middle|\; [T(v_2)]_\gamma \;\middle|\; \cdots \;\middle|\; [T(v_n)]_\gamma \,\right]}_{\text{the matrix representation of } T \text{ w.r.t. } \beta \text{ and } \gamma.}$$

where $[T(v_i)]_\gamma = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}$ if $T(v_i) = c_1 w_1 + c_2 w_2 + \cdots + c_m w_m$.

$$T \text{ s.t. } T(v_j) = \sum_{i=1}^{m} a_{ij} w_i \longleftarrow\!\!| \; A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

the matrix representation of $T$ w.r.t. $\beta$ and $\gamma$.

e.g. $V = P_3 = \{ p \in k[x] : \deg p \leq 3 \} = \mathrm{Span}_k \underline{\{1, x, x^2, x^3\}}$.

a basis $\beta$ of $V$.

$W = P_2 = \{ p \in k[x] : \deg p \leq 2 \} = \mathrm{Span}_k \underline{\{1, x, x^2\}}$.

$\gamma$. a basis of $W$.

enlarge $P_3$

$(T : P_3 \to P_2)$ : formal differentiation, the restriction of the unique linear map $T' : k[x] \to k[x]$ with $T'(x^n) = nx^{n-1}$ $\forall n \geq 0$.

$$[T]_\beta^\gamma = \left[ [T(1)]_\gamma \,\middle|\, [T(x)]_\gamma \,\middle|\, [T(x^2)]_\gamma \,\middle|\, [T(x^3)]_\gamma \right] = \left[ [0]_\gamma \,\middle|\, [1]_\gamma \,\middle|\, [2x]_\gamma \,\middle|\, [3x^2]_\gamma \right]$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

**Note:** $T'$ also restricts to a map $S : P^3 \to P^3$ given by formal diff; we have $[S]_\beta^\beta = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Now, for a v.s. $V$ and a fixed basis $\beta$ of $V$, taking $W = V$ in the above bijection gives a bijection

$$\mathrm{End}_k(V) = \{ T : V \to V \text{ linear} \}. \xleftrightarrow{[\ ]_\beta^\beta} M_{n\times n}(k) = M_n(k), \ n := \dim V.$$

$$\Phi : T \longmapsto [T]_\beta^\beta .$$

We know $\Phi$ is linear by linear algebra. (Check this!)

So $\Phi$ is a v.s. isomorphism.

In particular, $\mathrm{End}_k(V)$ is not comm if $n > 1$, and $\dim_k(\mathrm{End}_k(V)) = \dim M_n(k) = n^2$

Moreover, $\Phi$ respects mult because $[S' \circ S]_\beta^\beta = [S']_\beta^\beta \circ [S]_\beta^\beta$ !

So $\Phi$ is an algebra isomorphism (note that $[Id_V]_\beta^\beta = I_n$.)

**Ex.** Consider $T: \mathbb{R}^4 \to \mathbb{R}^3$, $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \mapsto \begin{bmatrix} 3x+w \\ y-z \\ x+y+2w \end{bmatrix}$.

This gives a linear map. Find its matrix w.r.t the standard bases $\beta = \{e_1, e_2, e_3, e_4\}$ and $\gamma = \{e_1, e_2, e_3\}$ of $\mathbb{R}^4$ and $\mathbb{R}^3$.

**Next time:** group algebras and path algebras