Math 4140. Lecture 3.

01. 22. 2021

Last time:

- Linear algebra review
- The free vector space $\Bbbk X$ on a set X, and its universal property.

Today.

- More on free vector spaces
- Definition of algebras
- First examples of algebras, group algebras

1. Free Vector Spaces.

Recall that for any set X, the v.s. $KX = \left\{ \sum_{\substack{x \in X \\ \text{finite} \\ \text{sum}}} c_x x \;\middle|\; c_x \in K \right\}$

has the following property. "universality"

Prop 1. Given any vector space W and any function $f: X \to W$,

there is a unique linear map $\bar{f} : KX \to W$ s.t. $f = \bar{f} \circ i$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"'] & W \\ KX \arrow[ur, dashed, "\bar{f}"'] \arrow[u, phantom, "\circ"] & \end{tikzcd}$$

where i is the injective map

$i: X \to KX$, $x \mapsto x$.

Ex. Do HW1.(7)-(8).

universal property of quotient gps.

Point: To get a linear map from KX to a v.s. W, it suffices to have a set map from X to W.

Remarks: Fix a field K.

(1). **Fact 1:** If V and V' are both free v.s. on a set X, then there is a unique isomorphism $\varphi: V \longrightarrow V'$ of v.s. Ex*: prove this.

(2) By Props. 1 & Fact 1., up to iso. we can say KX is the free vector space over X.

(2) Def 2 is an example of a def. by universal property.

We will use universal properties throughout the semester.

1. Definitions

K-algebras

Def 1. (Algebras (over K)) A K-algebra is a K-vector space $(A, +, \cdot)$ (scalar mult) equipped with an associative, bilinear and unital (binary) multiplication $m: V \times V \to V$. Here,

- associativity means $(ab)c = a(bc)$ $\xrightarrow[\text{for}]{\text{notation}}$ $m(m(a,b), c) = m(a, m(b,c))$
- bilinearity means $a(b+b') = ab + ab'$, $a(\lambda b) = \lambda(ab)$
  $(a+a')b = ab + a'b$, $(\lambda a)b = \lambda(ab)$ $\Big\}$ $\forall \lambda \in K$, $a, a', b, b' \in A$.

linearity in the first argument.

linearity in the second argument

- "unital" means m has a unit 1 with $a1 = a = 1a$ $\forall a \in A$.

Note: ("$\{\text{algebras}\} \subseteq \{\text{rings}\}$") So an K-algebra is always a ring. (highlighted : dist.)

More defs: Let A be a K-algebra.

Note (b). $(3v+4v')\cdot(v-v')$ $\quad v, v'$ basis elts

$= 3vv' + 4v'v - 3vv' - 4v'v'$

- The dimension of A is the dimension of A as a v.s. over K, i.e., $\dim_K A$.

- We say A is commutative if its multiplication (m) is commutative, i.e., if $ab = ba$ $\forall a, b \in A$.

- Let B be another algebra. Then an algebra homomorphism from A to B is a linear map $f: A \to B$ s.t. (1) $f(aa') = f(a)f(a')$ $\forall a, a' \in A$ and (2) $f(1_A) = 1_B$. $f(a) = f(1_A \cdot a) = f(1_A)\cdot f(a) \overset{?}{\Longrightarrow} f(1_A)$ not quite, need inverse for $f(a)$

Note: (Behavior of algebras are "controlled" by their bases.)

(a). To know the behavior of an algebra hom $f: A \to B$, it suffices to know the image of f on a basis of A (since f is linear). (b). To understand m, knowing $m(v, v')$ is enough.
v, v' basis elts

A comment on units:

- Not all books require units for rings or algebras, but we'll do.

- There's a mistake in E.H. Ex 1.3. See HW1.pdf.

Examples of algebras.

(1) (k/k) Any field k is itself a k-algebra.

v.s. ✓ m: usual multiplication. ✓ dim: $\dim_k k = 1$

(2) $(\mathbb{C}/\mathbb{R})$ $\mathbb{C}$ is a two-dimensional algebra over $k = \mathbb{R}$.
$\overset{\shortparallel}{k}$

v.s. ✓ basis: $\{1, i\}$ m: usual multiplication ✓

$\underset{\mathbb{C}}{\underline{a}}(\underset{\mathbb{R}}{\underline{\lambda}}\underset{\mathbb{C}}{\underline{b}}) = \lambda(ab)$ ← part of bilinearity.

(3) $(M_n(k)/k)$ The set of $n \times n$ matrices with entries in k is an algebra over k with the usual addition and multiplication. Ex: Carefully check the axioms.

(4). (polynomial algebras $k[x]$, $k[x,y]$, ...)

(5). (Endomorphism algebras $\mathrm{End}_k(V)$, V a k-v.s.)

next time!

Note: $\mathrm{End}_k(V) \cong M_n(k)$ if $\dim_k V = n$.