Math 4140. Lecture 2. 01. 20. 2021.

Last time: · Axiomatic definitions of groups, rings, fields and vector spaces.

Today. · Linear algebra review

· The free vector space on a set, via "universal properties"

1. Linear algebra review.

(1). We already saw that a vector space over a ground field k is defined as an abelian group $(V, +)$ equipped with a well-behaved map $k \times V \to V$, $(c, v) \mapsto c \cdot v$

satisfying properties (a)–(d) from the last page of Lecture 1.

You should be very comfortable with basic linear algebra: vectors, matrices, vector/matrix arithmetic, determinants, linear maps, bases, etc.

Eg. ① $(\mathbb{R}^n, +)$ forms a v.s. over $\mathbb{R}$,

e.g. $n=3$. $\quad \underset{\substack{c \in \mathbb{R} \\ \| \\ k}}{2} \cdot \underset{v \in V = \mathbb{R}^3}{\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}} = \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix}$

now scalars are the elts of $\mathbb{C}$. $\to a+bi$.

$a+bi = \underset{\lambda \in k}{(a+bi)} \cdot 1$

② $(k^n, +)$ is a v.s. over k. of course, $k=\mathbb{C}$ is also a vector space over $\mathbb{C}$, basis $=\{1\}$ $\dim_{\mathbb{C}} \mathbb{C} = 1$

③ $(\mathbb{C}, +)$ can be viewed as a vector space over $\mathbb{R}$

$\mathbb{C} = \{ a+bi \mid a, b \in \mathbb{R} \}$ $\quad i^2 = -1$ with basis $\{1, i\}$ $\longrightarrow \dim_{\mathbb{R}} \mathbb{C} = 2$

$a+bi = \underset{\lambda_1}{a} \cdot 1 + \underset{\lambda_2}{b} \cdot i$

④ $M_n(k)$, the set of $n \times n$ matrices over a field k, forms

$(+, \ )$ a vector space over k.

e.g. $M_2(\mathbb{R})$, typical elts: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$+$: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$

Scalar mult : $7 \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 14 \\ 21 & 28 \end{bmatrix}$.

In fact, we'll see that $M_n(k)$ is more: we can also multiply elts of $M_n(k)$ (not something you can always do for v.s. think $\mathbb{R}^3$), and the multiplication will actually make $M_n(k)$ an <u>algebra</u>.

More familiar notions: Fix a ground field k.

(1). Linear maps: A linear map between two k vector spaces V_1, V_2 is a map $f: V_1 \to V_2$ s.t.

$$\begin{cases} \text{(a)} \quad f(v_1+v_2) = f(v_1)+f(v_2) & \forall v_1, v_2 \in V_1 \\ \text{(b)} \quad f(c\cdot v) = c\cdot f(v) & \forall v \in V_1 \end{cases}$$

(2). Basis: A basis of a k-vector space V is a subset $B \subseteq V$ s.t.

$$\begin{cases} \text{(a) } B \text{ spans } V, \text{ i.e., every elt in } V \text{ is a lin. comb of elts. of } B. \\ \text{(b) } B \text{ is linearly independent: } \left.\begin{array}{l} c_1v_1+\dots+c_nv_n=0 \\ v_i \in B,\ c_i \in R \end{array}\right\} \Rightarrow c_1=\dots=c_n=0. \end{cases}$$

(3) Dimension:

Fact: Every v.s. has at least one basis, and all bases of it have the same size.

Def: The dimension of a v.s. V is the size of any of its bases.

(3)' (continued.) A v.s. is finite dimensional if it has a finite spanning set (equivalently, if the v.s. has a finite basis).

More on bases:

Recall that the behavior of a v.s. is often "controlled" by the behavior of any chosen basis of it:

E.g. (1). (decomp.) Let V be a v.s. and $B \subseteq V$ a basis of V. Then every elt $x \in V$ has a unique decomposition $x = \sum_{v \in B} c_v v$

(2) (linear map.) Let $f: V \to W$ be a lin. map and B a basis of V. Then the images of the basis vectors determine the behavior of f on V:

e.g. $x = \sum_{v \in B} c_v v \implies f(x) = f\left(\sum_{v \in B} c_v v\right) = \sum_{v \in B} f(c_v v) = \sum_{v \in B} c_v f(v)$

2. The free vector space on a set:

Def. $(X \to KX)$ Given a set X, we can define a v.s. KX as the set of linear combinations of elts of X, where we view X as a basis.

set v.s

e.g. $X = \{A, B\}$, $k = \mathbb{R}$, $\rightsquigarrow$ $KX = \{ c_1 A + c_2 B \mid c_1, c_2 \in \mathbb{R} \}$

$+$: $(c_1 A + c_2 B) + (d_1 A + d_2 B)$

$= (c_1 + d_1) A + (c_2 + d_2) B$

$\cdot$: $d \cdot (c_1 A + c_2 B) = (d c_1) A + (d c_2) B$.

Important: The set X is linearly ind and a basis by definition.

Def. (the free vector space on a set) Let X be a set.
A <u>free vector space</u> on X means a v.s. V equipped with an injective map $i: X \to V$ s.t.

for any vector space W and set map $f: X \to W$, there is a unique linear map $\bar{f}: V \to W$. s.t. $f = \bar{f} \circ i$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "i"'] & W \\ V \arrow[ur, dotted, "\exists ! \bar{f}"'] & \end{tikzcd}$$

$f: X \longrightarrow W$ (set theoretic)

$\exists ! \bar{f}$ linear map

Prop: KX is a free v.s. on X.

Pf: Take $i: X \to V = KX$ to be the map with
$i(x) = \underline{\underline{x}} \in KX$. $\forall x \in X$.

a basis elt in KX.

Now we need to show we can always find $\bar{f}$ from f.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"'] & W \\ KX \arrow[ur, dashed, "\exists!\,\bar f"'] & \end{tikzcd} \qquad x \mapsto f(x)$$

$\exists!\,\bar f$ linear map.

(1) We'll first assume we can find $\bar f$ and show that $\bar f$ is unique:

If we have $\bar f$, then $\bar f \circ i = f$.

So $\bar f \circ \underbrace{i(x)}_{x} = f(x)$ $\forall x \in X$, i.e. $\bar f(x) = f(x)$ $\forall x \in X$.

That is, if the desired $\bar f$ exists, then we know $\bar f(x)$ for all elts x in the basis X of $V = KX$. This means $\bar f$ is uniquely determined.

(2) We'll show we can find $\bar f$. Take $\bar f$ to be the map s.t.

$$(*) \quad \bar f\Big(\sum_{v \in X} c_v \cdot v\Big) = \sum_{v \in X} c_v f(v).$$

Ex. (*) defines a linear map with the desired property.