

Last time:

- Axiomatic definitions of groups, rings, fields and vector spaces.

Today:

- Linear algebra review
- The free vector space on a set, via "universal properties"

## 1. Linear algebra review.

(1). We already saw that a vector space <sup>over a ground field  $K$</sup>  is defined as an abelian group  $(V, +)$  equipped with a well-behaved map  $K \times V \rightarrow V, (c, v) \mapsto c \cdot v$

satisfying properties (a)-(d) from the last page of Lecture 1.

You should be very comfortable with basic linear algebra: vectors, matrices, vector/matrix arithmetic, determinants, linear maps, bases, etc.



E.g. ①  $(\mathbb{R}^n, +)$  forms a v.s. over  $\mathbb{R}$ ,

e.g.  $n=3$ .

$$c \in \mathbb{R} \quad \underbrace{\quad}_{\mathbb{K}} \quad v \in V = \mathbb{R}^3 \quad \rightarrow \quad 2 \cdot \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix}$$

now scalars are the  
elts of  $\mathbb{C} \rightarrow a+bi$ .  
 $a+bi = (a+bi) \cdot 1$   
 $\lambda \in \mathbb{K}$

②  $(\mathbb{K}^n, +)$  is a v.s. over  $\mathbb{K}$ . of course,  $\mathbb{K} = \mathbb{C} \Rightarrow$  also -

vector space over  $\mathbb{C}$ , basis:  $\{1\}$

③  $(\mathbb{C}, +)$  can be viewed as a vector space over  $\mathbb{R}$   $\dim_{\mathbb{R}} \mathbb{C} = 2$

$$\left\{ \begin{array}{l} a+bi \\ \text{"} \\ i^2 = -1 \end{array} \mid a, b \in \mathbb{R} \right\}$$

with basis  $\{1, i\} \rightarrow \dim_{\mathbb{R}} \mathbb{C} = 2$

$$a+bi = \frac{a}{\lambda_1} \cdot 1 + \frac{b}{\lambda_2} \cdot i$$

④  $M_n(k)$ , the set of  $n \times n$  matrices over a field  $k$ , forms  
(+,  $\cdot$ ) a vector space over  $k$ .

e.g.  $M_2(\mathbb{R})$ , typical elts:  $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$   $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$+ : \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$$

$$\text{Scalar mult: } 7 \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 14 \\ 21 & 28 \end{bmatrix}.$$

In fact, we'll see that  $M_n(k)$  is more: we can also multiply  
elts of  $M_n(k)$  (not something you can always do for v.s. think  $\mathbb{R}^3$ ),  
and the multiplication will actually make  $M_n(k)$  an algebra.

More familiar notions: Fix a ground field  $k$ .

(1). Linear maps: A linear map between two  $k$  vector spaces  $V_1, V_2$  is a map  $f: V_1 \rightarrow V_2$  s.t.

$$\begin{cases} \text{(a)} & f(v_1 + v_2) = f(v_1) + f(v_2) & \forall v_1, v_2 \in V_1 \\ \text{(b)} & f(c \cdot v) = c \cdot f(v) & \forall v \in V_1 \end{cases}$$

(2). Basis: A subset of a  $k$ -vector space  $V$  is a subset  $B \subseteq V$  s.t.

$$\begin{cases} \text{(a)} & B \text{ spans } V, \text{ i.e., every elt in } V \text{ is a lin. comb of elts. of } B. \\ \text{(b)} & B \text{ is linearly independent: } \left. \begin{array}{l} c_1 v_1 + \dots + c_n v_n = 0 \\ v_i \in B, c_i \in k \end{array} \right\} \Rightarrow c_1 = \dots = c_n = 0. \end{cases}$$

(3) Dimension:

Fact: Every v.s. has at least one basis, and all bases of it have the same size.

Def: The dimension of a v.s.  $V$  is the size of any of its bases.



(3) (continued.) A v.s. is finite dimensional if it has a finite spanning set (equivalently, if the v.s. has a finite basis).

More on bases:

Recall that the behavior of a v.s. is often "controlled" by the behavior of any chosen basis of it:

E.g. (1). (decomp.) Let  $V$  be a v.s. and  $B \subseteq V$  a basis of  $V$ . Then every  $x \in V$  has a unique decomposition  $x = \sum_{v \in B} c_v v$

(2) (linear map.) Let  $f: V \rightarrow W$  be a lin. map and  $B$  a basis of  $V$ .

Then the images of the basis vectors determine the behavior of  $f$  on  $V$ :

e.g.  $x = \sum_{v \in B} c_v v \implies \underline{f(x)} = f\left(\sum_{v \in B} c_v v\right) = \sum_{v \in B} f(c_v v) = \sum_{v \in B} c_v \underline{f(v)}$

## 2. The free vector space on a set :

Def. ( $X \rightarrow KX$ ) Given a set  $X$ , we can define a v.s.  $KX$  as the set of linear combinations of elts of  $X$ , where we view  $X$  as a basis.

eg.  $X = \{A, B\}$ .  $K = \mathbb{R}$ ,  $\rightarrow KX = \{c_1A + c_2B \mid c_1, c_2 \in \mathbb{R}\}$

$$+ : (c_1A + c_2B) + (d_1A + d_2B) = (c_1 + d_1)A + (c_2 + d_2)B$$

Important: The set  $X$  is linearly ind and a basis by definition.

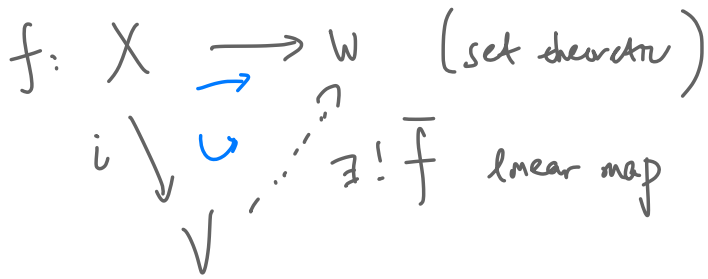
Def. (the free vector space on a set) Let  $X$  be a set.

A free vector space on  $X$  means a v.s.  $V$  equipped with an

injective map  $i: X \rightarrow V$  s.t.

for any vector space  $W$  and set map  $f: X \rightarrow W$ , there is

a unique linear map  $\bar{f}: V \rightarrow W$  s.t.  $f = \bar{f} \circ i$ .



Prop:  $KX$  is a free v.s. on  $X$ .

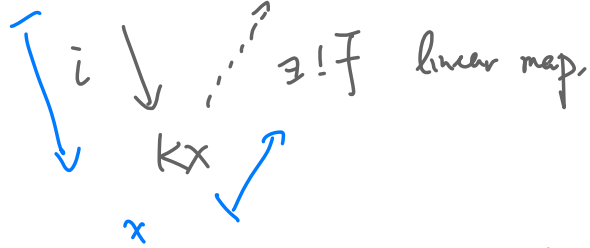
pf: Take  $i: X \rightarrow V = KX$  to be the map with

$$i(x) = \underline{x} \in KX. \quad \forall x \in X.$$

a basis elt in  $KX$ .

Now we need to show we can always find  $\bar{f}$  from  $f$ .

$$f: X \rightarrow W$$



(1) We'll first assume we can find  $\bar{f}$  and

show that  $\bar{f}$  is unique:

if we have  $\bar{f}$ , then  $\bar{f} \circ i = f$ .

so  $\bar{f} \circ i(x) = f(x) \quad \forall x \in X$ , i.e.  $\bar{f}(x) = f(x) \quad \forall x \in X$ .

That is, if the desired  $\bar{f}$  exists, then we know  $\bar{f}(x)$  for all elts  $x$  in the basis  $X$  of  $V = KX$ . This means  $\bar{f}$  is uniquely determined.

(2) We'll show we can find  $\bar{f}$ . Take  $\bar{f}$  to be the map s.t.

$$(*) \quad \bar{f} \left( \sum_{v \in X} c_v \cdot v \right) = \sum_{v \in X} c_v f(v)$$

Ex. (\*) defines a linear map with the desired property.