Math 4140 / 5140. Abstract Algebra II.

Course Information.

Instructor: Tianyuan Xu (Eddy)

Topic: Algebras and Representation Theory

Website: https://math.colorado.edu/~tixu6187/aa2.html

- has Canvas link
- lecture notes & HW posted under "LECTURES" tab

Office Hours: Mondays. 10–11 am. + appointments.

Grading: HW 30%, $\underbrace{\text{Midterm } 30\%, \text{ Final}}_{\text{take-home}}$ 40%

HW: — to be submitted on Canvas / Assignments, pdf only

— due on Wednesday nights 11:59 pm.

(the deadline is strict; no late submission possible)

— HW1 due on Jan 27.

— posted the previous Wed. Jan 20.

Textbook: "Algebras and Representation Theory" by Erdmann & Holm.

— free, available at CU library or Canvas / Files.

On to math ...

Today: Review: axiomatic def. of groups, rings, fields, and vector spaces; linear algebra

1. Groups

Def. A group (gp) is a pair $(G, \cdot)$ where G is a set and $\cdot$ is an operation s.t. (such that) the following holds:

(0). (Closure): $\cdot$ is a map $\cdot : G \times G \to G$. that is, $\forall a, b \in G$, there's an elt $a \cdot b$ in G (equiv., $\cdot$ is a binary operation on G.)

(1). (Associativity): $\forall a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(2). (Identity): G has an elt e s.t. $e \cdot a = a = a \cdot e$ $\forall a \in G$.

(3). (Inverse): $\forall a \in G$, $\exists b \in G$ s.t. $a \cdot b = e = b \cdot a$.

Recall that $(G, \cdot)$ is called abelian if $a \cdot b = b \cdot a$ $\forall a, b \in G$.

Examples: $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$ form abelian gps.

usual addition

$(\mathbb{N}, +)$ is not a gp since the inverse axiom fails.

$\mathbb{Z}_{\geq 0}$

important for us. $S_n \to$ symmetric gps $\quad A_n \to$ alternating gps

Remark: We'll assume familiarity with the basic notions and facts of gp theory: subgps, Lagrange's Thm, gp homomorphisms, Isomorphism Thms, gp actions, ...

Remark: (Additive notation vs multiplicative notation.)

Sometimes we denote the binary op. on a gp G by $\underbrace{+}$ → additive

→ $3+5=8$.

and sometimes by $\underbrace{\cdot,\ \times,\ \text{or juxtaposition } (ab)}$ → multiplicative.

e.g., S_n: $(12)\cdot(12) = (12)(12) = id$

In $\mathbb{Z}$, if we write $+$ as $*$, then $3*5=8$.

Questions?

2. Rings

Def: A <u>ring</u> is a triple $(R, +, \cdot)$ where R is a set s.t.

(1) $+$ is a binary op on R

(2) $(a+b)+c = a+(b+c)$ $\forall a,b,c \in R$

(3) $\exists$ an elt in R, denoted by 0, s.t.
$0+r = r = r+0$ $\forall r \in R$

(4) $\forall a \in R$, $\exists$ an elt b. s.t. $a+b = 0 = b+a$

(5) $a+b = b+a$ $\forall a,b \in R$

$\Leftrightarrow$ $(R,+)$ is an abelian gp

(6) $\cdot$ is a binary op on R

(7). $(a\cdot b)\cdot c = a\cdot(b\cdot c)$ $\forall a,b,c \in R$

on $\cdot$

"the unit"

(9). $\exists$ an elt in R, 1, s.t. $1 \neq 0$, $a\cdot 1 = a = 1\cdot a$ $\forall a \in R$.

interaction of $+$ and $\cdot$

(8). (Distributivity) $a\cdot(b+c) = ab + a\cdot c$, $(a+b)\cdot c = a\cdot c + b\cdot c$ $\forall a,b,c \in R$.

Rmk: (a) In our course, we alway include an axiom for our rings

(9). R has an elt, 1, s.t $1 \cdot a = a = a \cdot 1$ $\forall a \in R$.

In other words, we always assume our rings to be "unital".

(b) A ring R is commutative if $a \cdot b = b \cdot a$ $\forall a, b \in R$.

(c) One can show that (HW) in a (unital) ring R,

(i). $0 \cdot r = 0 = r \cdot 0$ $\forall r \in R$.

(ii). If $1 = 0$, then $R = \{0\}$, which is why we assume $1 \neq 0$

$r = r \cdot 1 = r \cdot 0 = 0$ $\forall r$

Examples: $(\mathbb{Z}, +, \cdot)$, $(\mathbb{Q}, +, \cdot)$, $(M_n(k), +, \cdot)$, ...

$n \times n$ matrices with entries in a field k

3. Fields

Def 1. A field is a triple $(F, +, \cdot)$ satisfying a familiar set of axioms (see Wikipedia / Calculus books.)

Def 2. A field is a triple $(F, +, \cdot)$ s.t. $(F, +)$ is an abelian gp, $(F \setminus \{0\}, \cdot)$ is an abelian gp and $\cdot$ distributes over $+$.

Def 3. A field is a commutative ring $(F, +, \cdot)$ where every nonzero elt is invertible multiplicatively.

$\forall a \in F \setminus \{0\}$, $\exists\, b$ s.t. $ab = 1 = ba$.

Exercise: Convince yourself that the three definitions are equivalent.

Examples:

$(\mathbb{Z}, +, \cdot)$? ✗. ......

$(\mathbb{R}, +, \cdot)$, $(\mathbb{Q}, +, \cdot)$ ✓.

$M_n(\mathbb{R})$, $n \geq 2$? ✗. → also, not every nonzero matrix is invertible.

for one thing, $\cdot$ is not commutative.

$GL_n(\mathbb{R})$, $n \geq 2$? ✗

4. Vector spaces

Let k be a field.

Def. A vector space over k is an abelian gp $(V, +)$ equipped with a map $\cdot : k \times V \to V$, $(c, v) \mapsto c \cdot v$ s.t.

"action of k on V by scaling"

a scalar multiple of v

(a). $1 \cdot u = u \quad \forall u \in V$

(b). $c \cdot (u + v) = c \cdot u + c \cdot v$

(c). $(c + d) \cdot u = c \cdot u + d \cdot u$

(d). $(c \cdot_k d) \cdot u = c \cdot (d \cdot u)$

$\forall c, d \in k$, $u, v \in V$.

"the action is well-behaved".

More review next time...