MATH 3140. HOMEWORK 9

due Wednesday, Oct. 30

(1) Let G be a group and let X be a set. In class we showed that the data of a group action $A : G \times X \to X$ is equivalent to the data of a group homomorphism $\Phi : G \to S_X$, where S_X is the symmetric group on X. We proved that a group action A naturally gives rise to a group homomorphism Φ_A, and described how a group homomorphism Φ naturally gives rise to a map A_Φ. Recall the definition of A_Φ, then prove that A_Φ is a group action.

(2) When proving Cayley's theorem we used the "obvious" fact that $G/\{1_G\} \cong G$ for any group G. Prove this fact.

(3) Consider the group $G = \{e, a, b, c\}$ with the following multiplication table (where the entry in Row x, Column y equals xy).

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

(a) Recall from the last homework that up to isomorphism $G_1 = \mathbb{Z}/4\mathbb{Z}$ and $G_2 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ are the only two groups of order 4. Which of G_1 and G_2 is G isomorphic to, and why?

(b) Relabel the elements e, a, b, c of G as $1, 2, 3, 4$, respectively, and identify S_G with S_4 (as we did for the symmetric group S_3 in class). Consider the action of G on itself by left translation and the corresponding homomorphism $\Phi : G \to S_4$. Write down $\Phi(b)$ in cycle notation.

(c) Now consider the action of G on itself by right translation, i.e., the action $G \times G \to G, (g, x) \mapsto xg^{-1}$. Let $\Psi : G \to S_4$ be the group homomorphism associated to this action. Write down $\Psi(g)$ in cycle notation for all $g \in G$.

(4) Let G be a group and let $H \leq G$. The left translation action of G on the set G/H gives rise to a homomorphism $\varphi : G \to S_{G/H}$. What elements form the kernel of φ? Your answer should be of the form "For $g \in G$, $g \in \operatorname{Ker} \varphi$ if and only if ...".

(5) Let $G = S_4$ and let $X = \{(12)(34), (13)(24), (14)(23)\} \subseteq G$.

(a) Explain why X is a conjugacy class in G.

(b) Since X is a conjugacy class in G, G acts on X by conjugation $(g, x) \mapsto gxg^{-1}$, so we may consider the associated group homomorphism $\Phi : S_4 \to S_X (\cong S_3)$. Prove that Φ is surjective. You may want to relabel the elements of X like in Problem 2(b).

(c) Use the first isomorphism theorem to compute $|\operatorname{Ker}\Phi|$.

(d) Prove that $\operatorname{Ker}\Phi = \{e\} \cup X$. (Try to do this using just facts about general groups, without computing any actual composition of permutations. Also, note that $\operatorname{Ker}\Phi \cong G$ where G is as in Problem 3!)