MATH 3140. HOMEWORK 8

due Wednesday, Oct. 23

(1) Let $\varphi : G_1 \to G_2$ be a group homomorphism. Prove that φ is injective if and only if $\operatorname{Ker}\varphi = \{1_{G_1}\}$. (This can be viewed as a statement dual to the statement that "φ is surjective if and only if $\operatorname{Im}\varphi = G_2$", which is true by the definition of surjective maps.)

(2) Let G be a cyclic group, and let g be a generator of G.

(a) Prove that the map $\varphi : \mathbb{Z} \to G$ defined by $\varphi(n) = g^n$ for all $n \in \mathbb{Z}$ is a group homomorphism.

(b) Prove that any infinite cyclic group G must be isomorphic to $\mathbb{Z}$.

(c) Prove that any finite cyclic group G must be isomorphic to $\mathbb{Z}/n\mathbb{Z}$ where $n = |G|$. (We have thus classified all cyclic groups).

(d) Prove that for each prime number p, all groups of order p are isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

(3) Show that $\mathbb{Z} \times \mathbb{Z}$ is not isomorphic to $\mathbb{Z}$.

(4) Let $G = \{e, a, b, c\}$ be a group of order 4, where e is the identity of G. Let $V = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

(a) Prove that either G is cyclic, or a, b, c all have order 2.

(b) Prove that if a, b, c all have order 2, then G is isomorphic to V. Write down an explicit isomorphism. (Hint: think about what the multiplication table of G can look like in view of Problem 6 of Homework 1.)

Note that combined with Problem 2(c), this problem shows that up to isomorphism, $\mathbb{Z}/4\mathbb{Z}$ and V are the only two groups of order 4.