Math 3140. Homework 7

due Wednesday, Oct. 16

(1) Determine if each of the following maps $f : A \to B$ is well-defined. Explain your reasoning.

(a) Let $A = \mathbb{Q}, B = \mathbb{Z}$. To define $f(q)$ for $q \in \mathbb{Q}$, write $q = m/n$ for integers m, n, then set $f(q) = n$.

(b) Let A be the collection of similarity classes of matrices in $\mathrm{Mat}_{2\times 2}(\mathbb{R})$, so that each element in A is a set consisting of matrices similar to a given matrix. Let $B = \mathbb{R}$. To define $f(C)$ for $C \in A$, pick a matrix $M \in C$, then set $f(C) = \det M$.

(c) Let A to be the set of conjugacy classes of a group G, and let $B = \mathbb{Z}_{>0} \cup \{\infty\}$. To define $f(C)$ for a conjugacy class in A, pick an element $g \in C$ and set $f(C) = |g|$.

(2) Let $n \geq 2$. In class we proved that $|A_n| = |S_n|/2$ by the first isomorphism theorem. Here's an alternative proof: Let B_n be the set of all odd permutations in S_n. Prove that the map $f : A_n \to B_n$ defined by $f(\sigma) = (12)(\sigma)$ is a bijection, then conclude (explain why) that $|A_n| = |S_n|/2$.

(3) The *third isomorphism theorem* states that if H, K are normal subgroups of a group G with $H \subseteq K$, then

$$\frac{G/H}{K/H} \cong G/K.$$

Prove the theorem.