Math 3140. Homework 6

due Wednesday, Oct. 9

(1) Determine whether each of the following maps $\varphi_i : \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ is a group homomorphism (as always, when we mention $\mathrm{GL}_n(\mathbb{R})$, we assume implicitly that the group operation is matrix multiplication). For each map, justify your answer by either invoking the appropriate facts from linear algebra or giving a counter-example involving only 2×2 matrices.

(a) $\varphi_1(A) = A^t$, the transpose of A, for all $A \in \mathrm{GL}_n(\mathbb{R})$.
(b) $\varphi_2(A) = A^{-1}$ for all $A \in \mathrm{GL}_n(\mathbb{R})$.
(c) $\varphi_3(A) = (A^{-1})^t$ for all $A \in \mathrm{GL}_n(\mathbb{R})$.
(d) $\varphi_4(A) = A^2$ for all $A \in \mathrm{GL}_n(\mathbb{R})$.

(2) Let $\varphi : G_1 \to G_2$ be a group isomorphism. Show that $|\varphi(a)| = |a|$ for all $a \in G_1$ (even if $|a| = \infty$).

(3) An *automorphism* of a group G is an isomorphism $\varphi : G \to G$.

(a) Prove that the $\mathrm{Aut}(G)$, the set of automorphisms of a group G, forms a group under composition.
(b) For each $g \in G$, let $\varphi_g : G \to G$ be the conjugation map defined by $\varphi_g(a) = gag^{-1}$ for all $a \in G$. Show that $\varphi_g \in \mathrm{Aut}(G)$ for all $g \in G$.
(c) Prove that the map $\Phi : G \to \mathrm{Aut}(G)$ given by $\Phi(g) = \varphi_g$ is a group homomorphism.
(d) The kernel of Φ is of course a normal subgroup of G. What is this group? (Hint: it's a notion we've mentioned in homework and in class.)

(4) Let $\varphi : G_1 \to G_2$ be a group homomorphism.

(a) Prove that for any subgroup $H \le G_1$, the set

$$\varphi(H) = \{\varphi(h) : h \in H\}$$

is a subgroup of G_2.

(b) Prove that for any subgroup $H' \le G_2$, the *pre-image*

$$\varphi^{-1}(H') := \{a \in G_1 : \varphi(a) \in H'\}$$

is a subgroup of G_1. Note that here φ^{-1} is not a map taking an element to an element, but a map taking a set to a set; it makes sense even if φ is not bijective.

The point of this exercise: the image of a subgroup via a homomorphism is always a subgroup, as is the pre-image of a subgroup via a homomorphism.