Math 3140. Homework 3

due Wednesday, Sept. 18

Note: Unless otherwise specified, you should justify your answer for each problem, even if it starts with "describe . . . " or "find . . . ".

(1) Let $n \in \mathbb{Z}_{\geq 3}$, let P_n be a regular n-gon in a plane, and label the vertices of P_n counter-clockwise by $1, 2, 3, \cdots, n$. Let D_n be the associated dihedral group—the group of symmetries of P_n.

(a) Let r be the counter-clockwise rotation by $2\pi/n$ in the plane P_n is in, and let s be the reflection in the plane across the perpendicular bisector of the edge $1-2$ of P_n. Explain why the list

$$1, r, r^2, \cdots, r^{n-1}, s, sr, sr^2, \cdots, sr^{n-1}$$

is a complete and non-redundant list of the elements of D_n.

(b) Prove that $rs = sr^{-1}$.

(c) Prove that D_n is not abelian.

(d) Let us call the expressions in the list from Part (a) the *standard forms* of the elements of D_n. Write down the multiplication table of D_4, where you represent each element in its standard form. You don't need to show your computation for this problem.

(e) Using their standard forms, describe the elements of order 2 in D_n for a general number n.

(2) Let G be a group and let $H \subseteq G$. Prove that H is a subgroup of G if and only if $ab^{-1} \in H$ for all $a, b \in H$.

(3) Let $n \in \mathbb{Z}_{\geq 2}$, let S_n be the symmetric group on n letters, and let

$$A_n = \{\sigma \in S_n : \sigma \text{ is even}\}.$$

(a) Prove that A_n is a subgroup of S_n. You may use the result of the previous problem if you want. (Here A_n is called an *alternating group*.)

(b) Find the elements of A_3.

(4) Let G be a group. The *center* of G is defined to be the set

$$C_G := \{a \in G : ab = ba \text{ for all } b \in G\}.$$

(a) Prove that C_G is a subgroup of G.

(b) Prove that C_G is abelian.

(5) Find the center of the dihedral group D_4.