MATH 3140. HOMEWORK 13

due Wednesday, Dec. 11.

Note: Throughout the following problems, assume R is a ring.

(1) Prove that if R is unital, then an invertible element in R has a unique inverse.

(2) Let $a, b \in R$. Prove that

  (a) $(-a)b = -(ab) = a(-b)$;

  (b) $(-a)(-b) = ab$.

(3) Let $n \in \mathbb{Z}_{\geq 1}$. We claimed in class that $M_n(R)$, the set of $n \times n$ matrices with entries from R, is also a ring with the "usual" addition and multiplication. Prove this claim. (Be careful not to make extra assumptions such as R is $\mathbb{R}$ or R is commutative.)

(4) Let I, J be ideals in R.

  (a) Show that the set $I + J := \{i + j : i \in I, j \in J\}$ is an ideal in R.

  (b) Show that the set $I \cap J$ is an ideal in R.

  (c) Suppose $R = \mathbb{Z}$ and $I = m\mathbb{Z}, J = n\mathbb{Z}$ for some nonzero integers m, n. What is $I + J$ and what is $I \cap J$?

(5) Let R be a unital ring. Recall that a ring homomorphism between two unital rings is required, by definition, to send unity to unity.

  (a) Prove that if a map $\varphi : \mathbb{Z} \to R$ is a ring homomorphism, then it is *necessarily* given by $\varphi(n) = n1$ for all $n \in \mathbb{Z}$.

  (b) Show that the map $\varphi : \mathbb{Z} \to R, n \mapsto n1$ is indeed (*sufficient* to be) a ring homomorphism. (Thus, every unital ring admits a homomorphism from $\mathbb{Z}$!)

(6) Determine if the following maps are ring homomorphisms.

  (a) The map $f : \mathbb{Z} \to \mathbb{Z}$ with $f(n) = 2n$ for all $n \in \mathbb{Z}$.

  (b) The "formal differentiation map" $d : \mathbb{R}[x] \to \mathbb{R}[x]$, i.e., the unique group homomorphism from $\mathbb{R}[x]$ to $\mathbb{R}[x]$ with $d(1) = 0$ and $d(x^n) = nx^{n-1}$ for all $n \in \mathbb{Z}_{\geq 1}$.

  (c) The determinant map $\det : M_2(\mathbb{R}) \to \mathbb{R}$ which takes the determinants of matrices in $M_2(\mathbb{R})$.

  (d) The map $\text{eval}_3 : R \to \mathbb{R}$, where R is the ring of all continuous functions of the form $f : \mathbb{R} \to \mathbb{R}$ and $\text{eval}_3(f) = f(3)$ for every such function f.

(7) Prove that $\mathbb{R}[x]/(x^2 - 1) \ncong \mathbb{C}$, where x is a variable over $\mathbb{R}$.