Math 3140. Homework 11

due Wednesday, Nov. 20

(1) Consider the symmetric group $G = S_4$.

(a) Find a 2-Sylow subgroup P of G.

(b) Find all subgroups of G which have order 2. For each such group H, find an element $g \in G$ such that $gHg^{-1} \subseteq P$, where P is the group you found in (a).

(c) Find two subgroups H_1, H_2 of G of order 3, then find an element $g \in G$ such that $gH_1g^{-1} = H_2$.

(d) Explain why the existence of H_1, H_2 implies that G has exactly four subgroups of order 3.

(2) Let G be a group of order pq where p, q are primes, $p < q$ and $p \nmid q - 1$.

(a) Prove that G has a unique subgroup P of order p and a unique subgroup Q of order q.

(b) Show that both P and Q are normal in G.

(c) Show that $P \cap Q = \{1\}$.

(d) State the second isomorphism theorem, then use it to deduce from (c) that $PQ = G$.

(e) Prove that $G \cong \mathbb{Z}/(pq)\mathbb{Z}$. Make sure you explain every step.

(3) Let G be a group, let p be a prime dividing $|G|$, and let P be a p-Sylow subgroup of G. Prove that the number of subgroups of G conjugate to P is not divisible by p.

(4) Let G be a finite group, and X a finite G-set.

(a) Prove that

$$|X| = |X^G| + \sum_{x_i} \frac{|G|}{|\mathrm{Stab}_G(x_i)|}$$

where $X^G = \{x \in X : g \cdot x = x \text{ for all } g \in G\}$ and the sum is over representatives of the G-orbits in X which have more than one elements, with one representative x_i from each orbit. (This recovers the class equation when we consider the conjugate action of G on itself.)

(b) Now suppose that G is a p-group for some prime p. Use the equation from (a) to prove that $|X| \equiv |X^G| (\mathrm{mod}\ p)$.