# Math 3140. Homework 10

## due Wednesday, Nov. 13

(1) Let $G$ be a group and let $X, Y$ be $G$-sets. We say a map $f : X \to Y$ is *$G$-equivariant* if

$$f(g \cdot x) = g \cdot f(x)$$

for all $x \in X$ and $g \in G$. Prove that for any $x \in X$, the bijection between $G/\mathrm{Stab}_G(x)$ and $\mathrm{Orb}_G(x)$ from the orbit-stabilizer theorem is in fact an $G$-equivariant bijection.

(2) Let $G$ be a group and let $X$ be a $G$-set.

(a) Prove that for any $g \in G$ and $x \in X$, we have $\mathrm{Stab}_G(g \cdot x) = g\mathrm{Stab}_G(x)g^{-1}$.

(b) Prove that for any conjugate elements $g, h \in G$, say with $g = aha^{-1}$, we have $\mathrm{Fix}(g) = a \cdot \mathrm{Fix}(h)$ where $a \cdot \mathrm{Fix}(h) = \{a \cdot x : x \in \mathrm{Fix}(h)\}$.

(3) Let $G$ be a group.

(a) Prove that for any subgroup $H$ of $G$ and any $g \in G$, the set $gHg^{-1}$ is another subgroup of $G$.

(b) Let $\mathcal{S}$ be the collection of subgroups of $G$. Prove that the map $G \times \mathcal{S} \to \mathcal{S}, (g, H) \mapsto gHg^{-1}$ defines an action of $G$ on $\mathcal{S}$. We call this action *conjugation on subgroups.*

(c) The stablizer of $H \leq G$ under the above conjugation action is called the *normalizer* of $H$ and denoted by $\mathrm{Norm}_G(H)$. Prove that $H \subseteq \mathrm{Norm}_G(H)$ for any $H \leq G$.

(d) Prove that the orbit of a subgroup $H$ has size 1 if and only if $H$ is normal in $G$.

(4) Compute the following numbers.

(a) The size of the conjugacy class of $\sigma = (142)(36)$ in $S_6$.

(b) The number of flags with 7 vertical stripes of equal width, each of color red, white, or blue. As in class, identify two flags if one of them can be flipped to obtain the other.

(c) The number of necklaces with 6 beads, of which 2 are red , 2 are white, and 2 are blue. Identify two necklaces if one of them can be rotated to obtain the other.

(d) The number of bracelets with 7 beads, of which 2 are red , 2 are white, and 3 are blue. Identify two bracelets if we can obtain one of them from the other by a sequence of rotations and flips.