Math 3140. Homework 1

due Wednesday, Sept. 4

(1) Determine if each of the following sets forms a group under the specified operation. If not, point out one group axiom that fails and why it fails.

  (a) The set of odd integers, under usual addition.
  (b) The set $\mathbb{Q}_{>0}$ of positive rational numbers, under usual multiplication.
  (c) The set $\mathbb{R}[t]$ of polnomials in one variable with real coefficients, under multiplication.
  (d) The set of all differentiable functions $f : \mathbb{R} \to \mathbb{R}$, under addition.
  (e) The set of permutations $\sigma : [n] \to [n]$ such that $\sigma(1) = 2$.

(2) Let $n \in \mathbb{Z}_{>0}$ and let

$$\mathrm{SL}_n(\mathbb{R}) = \{M \in \mathrm{GL}_n(\mathbb{R}) : \det M = 1\}.$$

Prove that $\mathrm{SL}_n(\mathbb{R})$ forms a group under multiplication. (It's called a *special linear group*.)

(3) Show that the set $\{5, 15, 25, 35\}$ forms a group under multiplication modulo 40. What is the identity element of this group?

(4) Let G be a group and let $a, b, c \in G$. Recall that we proved the "sock-shoe principle", which says that $(ab)^{-1} = b^{-1}a^{-1}$.

  (a) Prove that $(abc)^{-1} = c^{-1}b^{-1}a^{-1}$. (You can use the sock-shoe principle if you want.)
  (b) State the obvious genearalization of (b) to from 3 to n elements in G. Convince yourself that the statement is true.

(5) Let G be a group and let $a, b \in G$. Prove that $(ab)^2 = a^2b^2$ if and only if $ab = ba$.

(6) Let G be a group and let T be its multiplication table, with ab in Row a, Column b for all $a, b \in G$. (Of course, if G is infinite then the table is infinite and we can't really write it down, but we can still fill each table cell easily.)

  (a) What condition on T is equivalent to G being abelian?
  (b) Prove that, if we write out a row in T, then that row contains every element of G, and every element appears exactly once in the row.
  (c) Suppose G is a group with three elements e, a, b, where e is the identity. Show that if we order the rows and columns of T in the order e, a, b, then there is a unique possibility for T.